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Two-Dimensional Potential Problems Concerning a Single Closed Boundary

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VI. *Two-Dimensional Potential Problems concerning a Single Closed Boundary.*

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(*Communicated by* G. I. TAYLOR, *F.R.S.*)

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(1) *Introduction.*

Solutions of LAPLACE's equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (1.11)$$

are required in many branches of Applied Mathematics, such as hydrodynamics, electro- and magneto-statics, steady flow of heat or electricity, etc. The two-dimensional form of the equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad \ldots\ldots\ldots\ldots \quad (1.12)$$

has a general solution

$$V = f(x + \iota y) + F(x - \iota y), \quad \ldots\ldots\ldots \quad (1.21)$$

f and F being arbitrary functions of their complex arguments. In the applications, one function alone is usually sufficient, and it is customary to write

$$w = \phi + \iota\psi = f(z) \quad \ldots\ldots\ldots\ldots \quad (1.22)$$

with $z = x + \iota y$, when ϕ and ψ usually have each some physical significance. Moreover, in most cases, the boundary conditions which have to be satisfied either are, or can be reduced to, the prescription of the boundary values of ϕ or ψ, or of their derivatives.

The fact that a solution of (1.12) is obtained from functions of a complex variable implies a conformal representation of the $\phi - \psi$ plane on the $x - y$ plane, the resulting net on the $x - y$ plane being equipotentials and stream-lines (or their analogues). Methods based on conformal representation for solving one class of hydrodynamical problem, that of free stream-lines first examined by KIRCHOFF and HELMHOLTZ, have been developed into a technique which can be applied to any obstacle having, in the two-dimensional trace, straight boundaries, and some progress has been made with

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curved boundaries.* On the other hand, hydrodynamical problems not involving discontinuous flow, for special boundaries, have been treated by many writers, but the methods of obtaining the solutions for the various types of motion have usually been *ad hoc*, and there is no development of a technique here. Some general results have been obtained, notably by LEATHEM† who obtained a simple general formula for the potential-stream function due to the uniform translation of an obstacle. He has shown that the " impulse " of the motion is $2\pi\rho \times$ the coefficient of $1/z$ in the expansion of w valid near infinity. He also obtained, at some length and in somewhat cumbrous form, a general solution for the fluid motion due to the rotation of the cylinder. The present writer gave, in a short note‡ which seems to have been overlooked, a solution of this problem in the form of a definite integral. D. M. WRINCH§ has also considered the translation and rotation of a cylinder whose section is a curve of fairly general type.

In practically all the above-mentioned work a subsidiary complex variable (to be denoted throughout this paper by $\zeta = \xi + \iota\eta$) is used, and the region outside the boundary in the z-plane is conformally represented on a standard region in the ζ-plane, w being then determined as a function of ζ. The $z - \zeta$ relation has to satisfy purely geometrical conditions, and it appeared to the writer that the solution of all the ordinary boundary-problems is implicit in this relation ; his papers on the circular arc|| were his first attempt to develop a technique whereby the solutions could be obtained, by general methods, from the transformation formula.

Of recent years, the successes of the LANCHESTER-PRANDTL " circulation " theory in aerodynamics,¶ JOUKOWSKI's transformation of a circle into a wing section, and the tentative application of these ideas to the theory of hydraulic turbines,** have revived interest in hydrodynamics ;†† while CARTER, and COE and TAYLOR,‡‡ have recently used conformal transformation to solve important problems in the design of dynamo-electric machinery. The present paper is an attempt to see how far it is possible to express, not only the potential functions, but also dynamical quantities such as the " impulse," and resultant forces and moments due to the fluid pressure, in terms of the transformation, for a single closed cylindrical boundary in a medium extending to infinity. The outlook is primarily hydrodynamical, but other interpretations of the solutions obtained are mentioned from time to time.

* GREENHILL, 'Theory of a Stream-Line past a Plane Barrier.' Aeronautical Research Committee, R. & M., 19 (1911). Also Appendix (1917), and LEVY, 'Proc. Roy. Soc.,' A, vol. 92, p. 285 (1916).

† 'Phil. Mag.,' vol. 35, p. 119 (1918) ; 'Phil. Trans.,' A, vol. 215, p. 439 (1915).

‡ 'Phil. Mag.,' vol. 35, p. 500 (1918).

§ 'Phil. Mag.,' vol. 49, p. 240 (1925) ; 'Proc. Lond. Math. Soc.,' vol. 24, p. 455 (1925).

|| 'Phil. Mag.,' vol. 35, p. 396, and vol. 36, p. 273 (1918).

¶ ROUSE, 'Engineering,' vol. 117, p. 1 (1924).

** 'Engineering,' vol. 121, p. 98 (1926).

†† 'Engineering,' vol. 124, p. 277 (1927).

‡‡ CARTER, 'Inst. El. Eng. Proc.,' vol. 64, p. 1115 (1926) ; COE and TAYLOR, 'Phil. Mag.,' vol. 6, p. 100 (1928).

As the problems to be considered are all two-dimensional, we shall deal with a plane section only, and when forces, etc., are considered, they will be those acting per unit length perpendicular to the section; these conventions must be understood throughout.

(2) *The Standard Transformation.*

(2.1) *Choice of a Standard Region.*—For the purpose in hand, the most convenient region upon which the doubly-connected region outside the boundary is to be conformally represented is a half-plane, the boundary being transformed into the real axis. SCHWARZ has given a formula for transforming the region outside a rectilinear polygon into the interior of a circle, the boundary transforming into the circumference, and LEATHEM* has made a study of periodic conformal transformations in which the region outside the closed boundary is transformed into a semi-infinite rectangle, the boundary transforming into the finite side. As, however, much use will be made of the method of images, these are less convenient than a half-plane as standard regions, since the inverse points for the circle, and the infinite repetitions in the strips, make the analysis more complicated. We take, therefore, as the standard transformation, one which transforms the region outside a closed boundary (which may reduce to the two sides of straight, broken, or curved line) in the z-plane into the upper half of the ζ-plane, the boundary in the z-plane becoming the real axis in the ζ-plane. LEATHEM's periodic transformation is then obtained by the substitution $\zeta = \tan \frac{1}{2}t$ $(t = r + \iota s)$, for which we shall see both mathematical and physical reasons; $s = 0$ is then the boundary, and a plot of the curves $r =$ const. and $s =$ const. forms what may be called the "fundamental net" (see § 3.1).

(2.2) *Region Outside a Closed Rectilinear Polygon.*—Although it is deducible from SCHWARZ' result, we will establish the transformation for the region outside a closed rectilinear polygon *ab initio*, partly for the sake of completeness, and partly because it will show us some of the necessary properties of the general transformation.

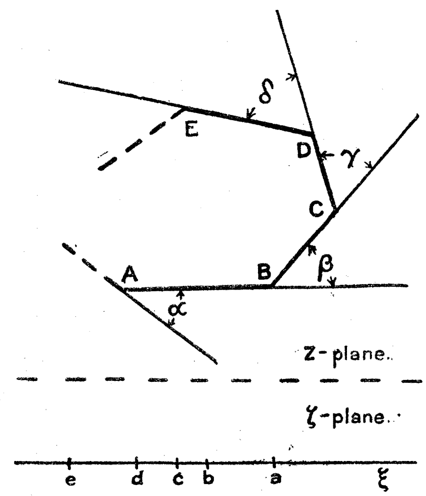

FIG. 1.

Let ABCD ... (fig. 1) be any finite, closed, rectilinear polygon in the z-plane, with exterior angles $\alpha, \beta, \gamma, \delta, \ldots$ (positive for a convex polygon, but negative at any re-entrant angle), and suppose it to be transformed into the upper half of the ζ-plane, the points $\zeta = a, b, c, d, \ldots$, on the real axis (so that $a, b, c, d, \ldots$, are all real) corresponding to the vertices A, B, C, D, ..., respectively. Then, by the usual argument (which we need not repeat) the corner A will require a factor $(\zeta - a)^{\alpha/\pi}$ in the

* 'Proc. R. Irish Acad.,' A, vol. 33, p. 35 (1916).

formula for $dz/d\zeta$. Taking account of all the corners, and assuming that ζ may come into the formula in other ways, we must have

$$dz/d\zeta = \mathrm{K} f(\zeta)\, \Pi\, (\zeta - a)^{\alpha/\pi}, \qquad (2.21)$$

where K is a complex constant determining the scale and orientation of the polygon, and $f(\zeta)$ is a function of ζ whose properties are still to be determined.

First, since the factors in $\Pi\,(\zeta - a)^{\alpha/\pi}$ take account of the changes of direction at the corners, it follows that $f(\zeta)$ must be real for real values of ζ, and also that it can have no zeroes for real, finite values of ζ. In order that the polygon may be closed, however, $\zeta = \infty$ must be a zero of $f(\zeta)$; and since $\Pi\,(\zeta - a)^{\alpha/\pi}$ is of degree $\Sigma\,\alpha/\pi = 2$ (unless $\zeta = \infty$ corresponds to a corner) this zero must be of order > 3. Again, any pole of $f(\zeta)$ must correspond to $z = \infty$, and as the correspondence between the upper half of the ζ-plane and the region of the z-plane outside the boundary must be one-one, it follows that $f(\zeta)$ must have one, and only one, pole in the upper half of the ζ-plane. Let this be at $\zeta = \xi_0 + \iota\eta_0$. Then, since $f(\zeta)$ is real for real values of ζ, $\xi_0 - \iota\eta_0$ must also be a pole of $f(\zeta)$, and these are the only two poles. Moreover, $f(\zeta)$ must have no zeroes, and no branch-points in the upper half of the ζ-plane, and consequently none in the lower half. Hence $f(\zeta)$ must be a rational, integral function of

$$1/(\zeta^2 - 2\zeta\xi_0 + \xi_0^{\,2} + \eta_0^{\,2}),$$

i.e.,

$$f(\zeta) = \frac{\mathrm{A}}{\zeta^2 - 2\xi_0\zeta + \xi_0^{\,2} + \eta_0^{\,2}} + \frac{\mathrm{B}}{(\zeta^2 - 2\xi_0\zeta + \xi_0^{\,2} + \eta_0^{\,2})^2} + \ldots, \qquad (2.22)$$

in which, by what has been said, A, B, ..., must all be real. Again, a small contour encircling $\zeta = \xi_0 + \iota\eta_0$ in the ζ-plane must transform into a curve at a great distance in the z-plane, described once only, so that we must have $\mathrm{A} = \mathrm{C} = \ldots = 0$, and thus

$$f(\zeta) = \frac{\mathrm{B}}{(\zeta^2 - 2\xi_0\zeta + \xi_0^{\,2} + \eta_0^{\,2})^2}. \qquad (2.23)$$

Absorbing the constant B in the constant K above, we have the required formula

$$\frac{dz}{d\zeta} = \frac{\mathrm{K}\Pi\,(\zeta - a)^{\alpha/\pi}}{(\zeta^2 - 2\xi_0\zeta + \xi_0^{\,2} + \eta_0^{\,2})^2}. \qquad (2.24)$$

It is convenient to standardise the position of the pole, and the most convenient point to adopt is evidently $\zeta = \iota$, so that the standard transformation formula will be taken in the form

$$\frac{dz}{d\zeta} = \frac{\mathrm{K}\Pi\,(\zeta - a)^{\alpha/\pi}}{(\zeta^2 + 1)^2}. \qquad (2.25)$$

In this formula there are $(n + 1)$ constants (n being the number of sides), and there are n conditions to be satisfied, which in general will be that the integrated result has prescribed values at the n vertices. There is thus one constant at our disposal. A better way of seeing this is to notice that any bi-linear transformation which transforms the real axis into itself, and for which $\zeta = \iota$ is a self-corresponding point will, while altering the values of $a, b, \ldots$, give another transformation of the same type for the same z-region. The formula for such a transformation is

$$\zeta = \frac{g\zeta' + h}{l\zeta' + m}, \qquad (2.26)$$

where

(a) g, h, l, and m must all be real, since the real axis is to transform into itself ; and

(b) $\zeta = \iota$ when $\zeta' = \iota$, so that

$$\iota = \frac{g\iota + h}{l\iota + m},$$

or

$$-l + m\iota = g\iota + h,$$

i.e.,

$$-l = h, \qquad g = m.$$

So

$$\zeta = \frac{g\zeta' + h}{-h\zeta' + g} = \frac{k\zeta' + 1}{k - \zeta'} \qquad (2.27)$$

where $k = g/h$ (and so k is real). There is thus a singly-infinite group of the required type of transformations ; that is, there is one disposable constant.

The particular transformation $\zeta = -1/\zeta'$ ($k = 0$) is important, as it enables us in cases of symmetry to determine some or all the constants $a, b, \ldots$, without integrating.

(2.3) *Extension to a Curved Boundary.*—Although it is possible to regard a curve as the limit of a polygon, and usefully so for many purposes, yet, for reasons given by LEATHEM,* such a conception is not likely to prove fruitful here. Other methods—unfortunately tentative—must be used. We may, however, deduce from the above that the general transformation formula must be of the form

$$\frac{dz}{d\zeta} = \frac{f(\zeta)}{(\zeta^2 + 1)^2} \qquad (2.31)$$

where $f(\zeta)$ (not to be confused with the $f(\zeta)$ of the last paragraph) is, in LEATHEM's terminology, a " curve-factor " of degree not exceeding 2, with no poles or zeroes in the upper half of the ζ-plane. We take this as the general transformation formula.

The occurrence of $(\zeta^2 + 1)$ in the denominator will suggest the substitution $\zeta = \tan \frac{1}{2}t$ as a mathematical expedient, as already mentioned. The physical meaning of this will appear later.

Integrated, the formula must lead to one of the type

$$z = \frac{g(\zeta)}{(\zeta^2 + 1)} + C, \qquad (2.32)$$

* *Loc. cit.*

C being the constant of integration, which can be assimilated into the formula, and will therefore be omitted. We must next examine the general properties of $f(\zeta)$ and $g(\zeta)$.

(2.4) *General Analytical Properties of* $f(\zeta)$ *and* $g(\zeta)$.—From the definition, we have

$$\frac{d}{d\zeta}\left(\frac{g(\zeta)}{\zeta^2+1}\right)=\frac{f(\zeta)}{(\zeta^2+1)^2}, \quad \ldots \quad (2.411)$$

or

$$(\zeta^2+1)\,g'(\zeta)-2\zeta g(\zeta)=f(\zeta). \quad \ldots \quad (2.412)$$

Differentiating this n times, we have

$$(\zeta^2+1)\,g^{(n+1)}(\zeta)+2(n-1)\,\zeta g^{(n)}(\zeta)+n(n-3)\,g^{(n-1)}(\zeta)=f^{(n)}(\zeta). \quad . \quad (2.42)$$

The values of f and g and their earlier derivatives, at the pole $\zeta=\iota$, and relations between these, will be required. Putting $\zeta=\iota$ in (2.42) for successive values of n, we derive

$$-2\iota g(\iota)=f(\iota), \qquad -2g(\iota)=f'(\iota),$$

and, from these two,

$$\iota f'(\iota)=f(\iota),$$

for which we shall soon see other reasons. Again,

$$2\iota g''(\iota)-2g'(\iota)=f''(\iota), \qquad 4\iota g'''(\iota)=f'''(\iota),$$

and so on.

The behaviour of z and $dz/d\zeta$ in the neighbourhood of the pole will also be required. Putting $\zeta=\iota+\delta$, we have

$$\frac{dz}{d\zeta}=\frac{dz}{d\delta}=\frac{f(\iota+\delta)}{\{(\iota+\delta)^2+1\}^2}=\frac{f(\iota+\delta)}{(2\iota\delta+\delta^2)^2}=\frac{F_{-2}}{\delta^2}+\frac{F_{-1}}{\delta}+F_0+F_1\delta+F_2\delta^2+\ldots, \quad \ldots \quad (2.431)$$

where

$$\left.\begin{aligned} F_{-2}&=-\tfrac{1}{4}f(\iota), \qquad F_{-1}=-\tfrac{1}{4}\{\iota f(\iota)+f'(\iota)\}\\ F_0&=\tfrac{1}{16}\{3f(\iota)-4\iota f'(\iota)-2f''(\iota)\}\\ F_1&=\tfrac{1}{48}\{6\iota f(\iota)+9f'(\iota)-6\iota f''(\iota)-2f'''(\iota)\}\\ &\text{etc.}\end{aligned}\right\} \quad \ldots \quad (2.432)$$

Integrating,

$$z=-\frac{F_{-2}}{\delta}+F_{-1}\log\delta+G_0+F_0\delta+\tfrac{1}{2}F_1\delta^2+\tfrac{1}{3}F_2\delta^3+\ldots, \quad . \quad (2.433)$$

G_0 being the constant of integration. Now z is to be a single-valued function of ζ (and therefore of δ), in the relevant region, so that we must have

$$F_{-1}=0 \quad \text{or} \quad \iota f(\iota)+f'(\iota)=0, \quad \ldots \quad (2.434)$$

a condition obtained above, the meaning and necessity of which is now evident. In virtue of this relation, it is possible to formally simplify the expressions for F_0, F_1, F_2, ... ; to do so is hardly worth while, except perhaps in the case of F_0 which becomes

$$F_0 = -\tfrac{1}{16}\{f(\iota) + 2f''(\iota)\}. \qquad (2.435)$$

Our expansions are, then,

$$\frac{dz}{d\zeta} = \frac{F_{-2}}{\delta^2} + F_0 + F_1\delta + F_2\delta^2 + \dots, \qquad (2.436)$$

$$z = -\frac{F_{-2}}{\delta} + G_0 + F_0\delta + \tfrac{1}{2}F_1\delta^2 + \tfrac{1}{3}F_2\delta^3 + \dots \qquad (2.437)$$

The latter might, for consistency of notation, be written

$$z = \frac{G_{-1}}{\delta} + G_0 + G_1\delta + G_2\delta^2 + G_3\delta^3 + \dots, \qquad (2.438)$$

with the evident relations between the F's and the G's.

Applying (2.434) to the polygon formula, (2.25), we find

$$\Sigma\frac{\alpha}{\iota - a} = -\iota. \qquad (2.441)$$

Splitting this up into its real and imaginary parts gives two necessary conditions to be satisfied by a, b, c, ... ,

$$\Sigma\frac{\alpha}{a^2+1} = 1, \qquad \Sigma\frac{a\alpha}{a^2+1} = 0. \qquad (2.442)$$

We shall also need the expansion of $z \div dz/d\zeta$ valid in the neighbourhood of the pole. Using (2.436) and (2.437) it will be found to be

$$\frac{z}{dz/d\zeta} = -\delta\left\{1 - \frac{G_0}{F_{-2}}\delta - \frac{2F_0}{F_{-2}}\delta^2 \dots\right\}. \qquad (2.45)$$

It has already been pointed out that, unless $\zeta = \infty$ corresponds to a corner of the boundary, $f(\zeta)$ is of degree 2. It is also evident that unless $\zeta = \infty$ makes $z = 0$, $g(\zeta)$ will also be of degree 2. When $\zeta = \infty$ corresponds to a corner, with exterior angle α_∞, then, by putting $\zeta = -1/\zeta'$, we must have

$$\frac{dz}{d\zeta'} = (\zeta')^{\alpha_\infty/\pi}\chi(\zeta'),$$

where $\chi(\zeta')$ is finite when $\zeta' = 0$. Consequently

$$\zeta^2\frac{dz}{d\zeta} = \zeta^{-\alpha_\infty/\pi}\chi(-1/\zeta),$$

or, as $\zeta \to \infty$,

$$\frac{dz}{d\zeta} \to \mathrm{K}\zeta^{-(2+\alpha_\infty/\pi)},$$

K being finite, *i.e.*,

$$\frac{f(\zeta)}{(\zeta^2+1)^2} \to \mathrm{K}\zeta^{-(2+\alpha_\infty/\pi)},$$

or

$$f(\zeta) \to \mathrm{K}\zeta^{2-\alpha_\infty/\pi}. \quad \ldots \ldots \ldots \ldots \quad (2.461)$$

In particular, if $\zeta = \infty$ corresponds to a cusp,

$$f(\zeta) \to \mathrm{K}\zeta. \quad \ldots \ldots \ldots \ldots \quad (2.462)$$

(2.5) *Known and Calculable Cases. Flat Plate.*—The simplest case is when the polygon reduces to the two sides of finite straight line, which is the trace of a flat plate.

We take the z-boundary to be the two sides of the line AB, from $z = c$ to $z = -c$ (fig. 2). If we let $\zeta = 0$ correspond to A, $\zeta = \infty$ must evidently correspond to B, *i.e.*, the negative half of the ξ-axis corresponds to the upper side of AB, and the positive half to the lower side. The exterior angles at A and B are each π, so (2.25) becomes

$$\frac{dz}{d\zeta} = \frac{\mathrm{K}\zeta}{(\zeta^2+1)^2}. \quad \ldots \quad (2.511)$$

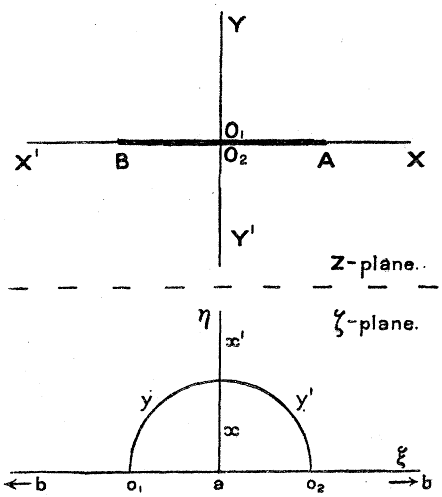

Fig. 2.

whence

$$z = -\frac{\mathrm{K}}{2(\zeta^2+1)} + \mathrm{C}.$$

When

$\zeta = 0, \quad z = c, \quad$ therefore $\quad c = -\tfrac{1}{2}\mathrm{K} + \mathrm{C},$

$\zeta = \infty, \quad z = -c, \quad$ therefore $\quad -c = 0 + \mathrm{C},$

i.e., $\mathrm{C} = -c$ and $\mathrm{K} = -2c$.

Inserting these values and reducing, we find

$$z = c\,\frac{(1-\zeta^2)}{(1+\zeta^2)}. \quad \ldots \ldots \ldots \ldots \quad (2.512)$$

In terms of t ($\zeta = \tan \tfrac{1}{2}t$) this becomes

$$z = c \cos t, \quad \ldots \ldots \ldots \ldots \quad (2.513)$$

the well-known formula.

Alternatively, if we let $\zeta = 0$ correspond to the point $z = 0$ on the upper side, symmetry shows that $b = -a$ and again $\alpha = \pi = \beta$, so the transformation formula is

$$\frac{dz}{d\zeta} = \frac{\mathrm{K}(\zeta - a)(\zeta + a)}{(\zeta^2+1)^2}. \quad \ldots \ldots \ldots \ldots \quad (2.514)$$

If we now use $\zeta = -1/\zeta'$, we find

$$\zeta'^2 \frac{dz}{d\zeta'} = \frac{K(-1/\zeta' - a)(-1/\zeta' + a)}{\{(1/\zeta')^2 + 1\}^2}$$

or

$$\frac{dz}{d\zeta'} = \frac{K(a\zeta' + 1)(a\zeta' - 1)}{(\zeta'^2 + 1)^2},$$

and as this must be the same as (2.514), therefore

$$a = 1 \quad \text{or} \quad -1.$$

Hence

$$\frac{dz}{d\zeta} = \frac{K(\zeta^2 - 1)}{(\zeta^2 + 1)^2}, \qquad (2.515)$$

and, integrating,

$$z = -\frac{K\zeta}{(\zeta^2 + 1)} + D.$$

When

$$\zeta = 0, \quad z = 0, \quad \text{therefore} \quad D = 0;$$

$$\zeta = 1, \quad z = c, \quad \text{therefore} \quad c = -\tfrac{1}{2}K.$$

So

$$z = \frac{2c\zeta}{\zeta^2 + 1}, \qquad (2.516)$$

or

$$z = c \sin t. \qquad (2.517)$$

Intersecting Planes.—The electrification of, and the flow past, two intersecting planes has been considered by MORTON.* We may derive the equivalent transformation by the above. Taking the z-boundary as ABCDA (fig. 3), let $\zeta = 0$ correspond to A, and denote $\underline{/DAE} = \alpha$ by $n\pi$. Then

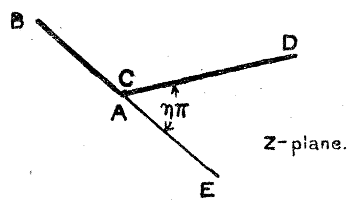

FIG. 3.

$$\beta = \pi, \quad \gamma = -n\pi, \quad \delta = \pi,$$

so

$$\frac{dz}{d\zeta} = K \frac{\zeta^n (\zeta - b)(\zeta - c)^{-n}(\zeta - d)}{(\zeta^2 + 1)^2}. \qquad (2.521)$$

Applying (2.442) to this, we must have

$$\frac{b}{b^2 + 1} - \frac{nc}{c^2 + 1} + \frac{d}{d^2 + 1} = 0,$$

$$n + \frac{1}{b^2 + 1} - \frac{n}{c^2 + 1} + \frac{1}{d^2 + 1} = 1,$$

which are equivalent to the conditions found by MORTON, that the integral should have no logarithmic term.

* 'Phil. Mag.,' vol. 1, p. 337, and vol. 2, p. 900 (1926).

Cross Lamina.—If the polygon is a pair of mutually bisecting, equal, perpendicular straight lines (fig. 4) and we take $\zeta = 0$ corresponding to A, then, by symmetry, $\zeta = \infty$ corresponds to E. The reciprocal transformation leads to the conclusions that $c = 1$ and $d = 1/b$; symmetry shows that $h = -b$, $g = -c = -1$, $f = -d = -1/b$. The transformation formula is thus,

$$\frac{dz}{d\zeta} = \mathrm{K}\,\frac{\zeta\,(\zeta^2 - 1)}{(\zeta^2 - b^2)^{\frac{1}{2}}\,(\zeta^2 - 1/b^2)^{\frac{1}{2}}\,(\zeta^2 + 1)^2}. \quad (2.531)$$

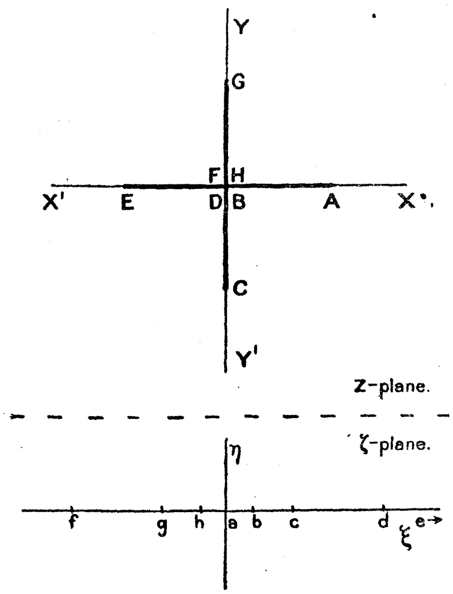

FIG. 4.

This will hold for any pair of mutually bisecting perpendicular straight lines, the value of b depending upon their ratio. When they are equal, the t transformation shows that each straight portion must correspond to an interval $\frac{1}{4}\pi$ of r, and so $b = \tan \frac{1}{8}\pi$. The integral of (2.531) can then be written

$$z = \mathrm{K}' \sqrt{\cos 2t} + \mathrm{C}. \quad . \quad . \quad (2.532)$$

If the origin in the z-plane is taken at the centre of the cross, and BA, of length c, along the x-axis, $\mathrm{K}' = c$ and $\mathrm{C} = 0$.

If there are n equal, equally spaced, rays, the transformation formula is

$$z = c\,(\cos \tfrac{1}{2}nt)^{2/n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.533)$$

Evidently, we might also have

$$z = c\,(\sin \tfrac{1}{2}nt)^{2/n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.534)$$

The transformation for a number of unequal, unequally spaced rays could also be obtained, but we refrain from giving details.

Regular Polygon.—We next consider the case of the regular polygon. If n is the number of sides, the exterior angles are all $2\pi/n$. Again, taking $\zeta = 0$ to correspond to a vertex, the t-transformation shows that the values of $a, b, c, \ldots$, are 0, $\tan \pi/n$, $\tan 2\pi/n$, $\ldots$, $\tan (n-1)\,\pi/n$; also $\tan (n-r)\,\pi/n = -\tan r\pi/n$. There is a slight difference in the ζ-formula according as n is odd or even, since in the latter case, $\zeta = \infty$ corresponds to a vertex also. Thus

$$\left.\begin{aligned} \frac{dz}{d\zeta} &= \mathrm{K}\,\frac{\left\{\zeta \prod_{r=1}^{r=\frac{1}{2}n+1} (\zeta^2 - \tan^2 r\pi/n)\right\}^{2/n}}{(\zeta^2+1)^2} \quad (n \text{ even}) \\ \text{or} \qquad & \mathrm{K}\,\frac{\left\{\zeta \prod_{r=1}^{r=(n-1)/2} (\zeta^2 - \tan^2 r\pi/n)\right\}^{2/n}}{(\zeta^2+1)^2} \quad (n \text{ odd}) \end{aligned}\right\} . \quad . \quad . \quad (2.541)$$

By the t-transformation, each of these reduces to the same form,

$$\frac{dz}{dt} = \mathrm{K}' (\sin \tfrac{1}{2}nt)^{2/n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.542)$$

which is equivalent to a formula given by LEATHEM.* The sine may be replaced by the cosine, in which case $\zeta = 0$ corresponds to the mid-point of a side. It is interesting to note the connection between this and (2.533); and, of course, § 2.51 is a detailed consideration of the simplest case, $n = 2$.

The integration cannot be effected algebraically except when $n = 2$; when $n = 4$ (square) elliptic functions to modulus $\tan^2 \frac{1}{8}\pi$ are needed—though by further transformation, the result is expressible in terms of functions with modulus $\frac{1}{2}\sqrt{2}$ (modular angle 45°). The integration may, however, be effected by series, and for this the cosine form is preferable. The relevant region in the z-plane corresponds to positive values of s, and as $s \to \infty$, $e^{\iota t} = e^{\iota r - s} \to 0$, while $e^{-\iota t} \to \infty$. We therefore write the transformation formula

$$\frac{dz}{dt} = \mathrm{K}\left\{\frac{e^{-\frac{1}{2}\iota nt} + e^{\frac{1}{2}\iota nt}}{2}\right\}^{2/n} = 2^{-2/n}\,\mathrm{K}e^{-\iota t}(1 + e^{\iota nt})^{2/n}. \quad . \quad . \quad . \quad . \quad . \qquad (2.543)$$

Now if $s > 0$ we may expand by the binomial theorem and integrate term by term, obtaining

$$z = \mathrm{C} + 2^{-2/n}\,\iota\mathrm{KF}\left\{-\frac{2}{n}, -\frac{1}{n}; \frac{n-1}{n}; -e^{\iota nt}\right\}, \quad . \quad . \quad . \quad . \quad . \qquad (2.544)$$

a series which converges quite rapidly for quite small values of n and s, and from which the "fundamental net" (see § 3.1) can be speedily plotted.

Rectangle.—The rectangle (fig. 5) will be briefly dealt with. By symmetry $d = -a$, and $c = -b$; by the reciprocal transformation, $b = 1/a$. So

$$\frac{dz}{d\zeta} = \mathrm{K}\frac{\sqrt{(\zeta^2 - a^2)(\zeta^2 - 1/a^2)}}{(\zeta^2 + 1)^2}. \qquad (2.551)$$

This can be integrated in terms of Jacobian elliptic functions to modulus a^2, and the ratio of the sides of the rectangle is determined in terms of the complete elliptic integrals to this and the complementary modulus.

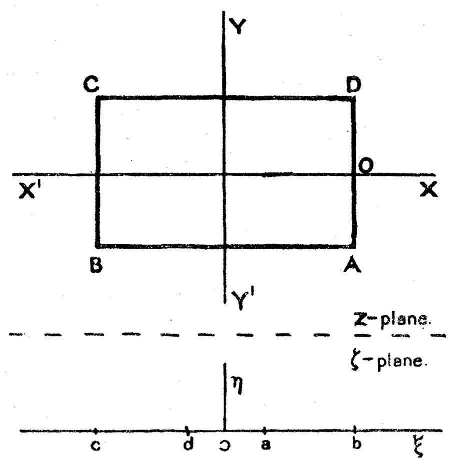

FIG. 5.

Details for the square, equilateral triangle, and rectangle have been worked out, including a skeleton table of elliptic functions to modulus $\tan^2 \frac{1}{8}\pi = (\sqrt{2} - 1)^2$, and the complementary modulus, and it is hoped to publish some of the results in the near future.

* 'Phil. Trans.,' A, vol. 215, pp. 439–487 (1915).

Circle.—By making n infinite in (2.544), we have the formula for the circle, which may be written

$$z = \iota K' e^{-\iota t} = -\iota K' \frac{\zeta + \iota}{\zeta - \iota} = K'' \frac{(\zeta + \iota)^2}{\zeta^2 + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.561)$$

and K″ may be identified with the radius. From this,

$$\frac{dz}{d\zeta} = -2\iota K'' \frac{(\zeta + \iota)^2}{(\zeta^2 + 1)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.562)$$

Ellipse.—It is known that the curves $s =$ const. in (2.513) are ellipses, so that if $s = s_0$ be taken as the boundary, we shall have

$$z = c \cos (t + \iota s_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.571)$$

or, in terms of ζ,

$$z = c \frac{(1 - \zeta^2) \cosh s_0 - 2\iota\zeta \sinh s_0}{(\zeta^2 + 1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.572)$$

Circular Arc.—The transformation for the circular arc was given by the author in a previous paper, and also independently by RICHMOND.* The t formula is, r being the radius, and 2α the angle subtended at the centre

$$z = -\iota r \frac{1 + \sin \frac{1}{2}\alpha \, . \, e^{-\iota t}}{1 + \sin \frac{1}{2}\alpha \, . \, e^{\iota t}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.581)$$

whence

$$z = -\iota r \frac{(\zeta + \iota)(\zeta - a\iota)}{(\zeta - \iota)(\zeta + a\iota)}, \qquad a = \frac{1 + \sin \frac{1}{2}\alpha}{1 - \sin \frac{1}{2}\alpha} \quad . \quad . \quad . \quad . \quad . \quad (2.582)$$

(where ζ has the meaning of the present, and not of the previous, paper).

Other Cases.—The transformation of the JOUKOWSKI aerofoil into a circle can be carried a step further—if desirable—by the use of (2.561).

D. M. WRINCH† has studied a family of curves for which the transformation formula is

$$z = nae^{-\iota t} + be^{n\iota t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.591)$$

and both she, and LEATHEM, have considered the more general transformation

$$z = ae^{-\iota t} + be^{\iota t} + ce^{2\iota t} + \ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.592)$$

(2.6) *Auxiliary Functions.*—There are also certain other functions of ζ, connected with, and derivable from, the transformation formula, that will be needed in the sequel. It seems more convenient to obtain them here than to interrupt the course of the argument later.

* 'Proc. Lond. Math. Soc.,' vol. 22, p. 483 (1924).

† 'Phil. Mag.,' vol. 48, pp. 692, 1089 (1924).

The first is a function of ζ that reduces, when ζ is real, that is, on the boundary, to the conjugate of z. On the boundary

$$z = x + \iota y = g(\xi)/(\xi^2+1) = \{g_x(\xi) + \iota g_y(\xi)\}/(\xi^2+1)$$

or

$$x = g_x(\xi)/(\xi^2+1) \quad \text{and} \quad y = g_y(\xi)/(\xi^2+1), \quad \ldots \ldots \quad (2.611)$$

where $g_x(\xi)$ and $g_y(\xi)$ are the real and imaginary parts of $g(\xi)$ (when ξ is real). Thus

$$x - \iota y = \{g_x(\xi) - \iota g_y(\xi)\}/(\xi^2+1), \quad \ldots \ldots \quad (2.612)$$

so that the function we seek is

$$(\bar{z})^* = \{g_x(\zeta) - \iota g_y(\zeta)\}/(\zeta^2+1), \quad \ldots \ldots \quad (2.613)$$

where the brackets round $\bar{z}$ denote that it is only on the boundary that this function is $\bar{z}$. Now if

$$\left.\begin{aligned} g_x(\zeta) &= g_x(\xi+\iota\eta) = g_{x1}(\xi,\eta) + \iota g_{x2}(\xi,\eta) \\ g_y(\zeta) &= g_y(\xi+\iota\eta) = g_{y1}(\xi,\eta) + \iota g_{y2}(\xi,\eta) \end{aligned}\right\}, \quad \ldots \ldots \quad (2.614)$$

and

then

$$g_x(\zeta) - \iota g_y(\zeta) = (g_{x1} + g_{y2}) + \iota(g_{x2} - g_{y1}). \quad \ldots \ldots \quad (2.615)$$

Also

$$\begin{aligned} g(\xi - \iota\eta) &= g_x(\xi-\iota\eta) + \iota g_y(\xi - \iota\eta) \\ &= g_{x1} - \iota g_{x2} + \iota(g_{y1} - \iota g_{y2}) \\ &= (g_{x1} + g_{y2}) - \iota(g_{x2} - g_{y1}). \quad \ldots \ldots \quad (2.616) \end{aligned}$$

Comparing (2.615) and (2.616), we see that

$$g_x(\zeta) - \iota g_y(\zeta) = \text{the conjugate of } g(\xi - \iota\eta),$$

which we shall denote by $\overline{g(\bar{\zeta})}$ (each bar denoting that the conjugate quantity is to be taken, and it being understood that $g(\xi - \iota\eta)$ is *first* calculated, and then its conjugate taken). Thus

$$(\bar{z}) = \overline{g(\bar{\zeta})}/(\zeta^2+1), \quad \ldots \ldots \quad (2.617)$$

$(\bar{z})$ reducing to $\bar{z}$ on the boundary.

In the same way, we have as the function that reduces to $d\bar{z}/d\xi$ on the boundary,

$$d(\bar{z})/d\zeta = \overline{f(\bar{\zeta})}/(\zeta^2+1)^2. \quad \ldots \ldots \quad (2.618)$$

It is to be noted, however, that $\overline{g(\bar{\zeta})}$, and $\overline{f(\bar{\zeta})}$ (which are by definition functions of ζ) will not, in general, be free from singularities in the upper half of the ζ-plane for $g(\zeta)$ may have poles or branch-points in the lower half of the ζ-plane, and then $g(\bar{\zeta})$ will

* We use the bar to denote conjugate quantities.

have poles or branch-points at the conjugate points, *i.e.*, in the upper half of the ζ-plane, and so, therefore, will $\overline{f(\bar{\zeta})}$. In particular, if, near the nth pole, $\xi_n - \iota\eta_n$, $g(\zeta)$ has a simple pole, the expansion of which in the neighbourhood is, with $\zeta = \xi_n - \iota\eta_n + \delta$

$$g(\zeta) = R_{-1n}/\delta + R_{0n} + R_{1n}\delta + \ldots,$$

then, near $\xi_n + \iota\eta_n$, with $\zeta = \xi_n + \iota\eta_n + \delta$

$$\overline{g(\bar{\zeta})} = \bar{R}_{-1n}/\delta + \bar{R}_{0n} + \bar{R}_{1n}\delta + \ldots \qquad (2.619)$$

Another function that will be needed is one that reduces, on the boundary, to the square of the modulus of z, *i.e.*, to the square of the radius vector in the z-plane from the origin to the corresponding point on the boundary. Now $g(\zeta)\,\overline{g(\bar{\zeta})}/(\zeta^2+1)^2$ evidently reduces to $z\bar{z} = r^2$ on the boundary, and so is the required function. It will be convenient to write

$$g(\zeta)\,\overline{g(\bar{\zeta})} = h(\zeta), \qquad (2.621)$$

and then

$$(r^2) = h(\zeta)/(\zeta^2+1)^2, \qquad (2.622)$$

the brackets denoting as before that the function is not everywhere equal to r^2, but only on the boundary. Alternatively, of course,

$$h(\zeta) = \{g_x(\zeta)\}^2 + \{g_y(\zeta)\}^2. \qquad (2.623)$$

The singularities of $h(\zeta)$ in the upper half of the ζ-plane are evidently those of $\overline{g(\bar{\zeta})}$, and expansions near the poles can be obtained by the use of (2.619). In some cases, such as that of (2.532), where $g(\zeta)$ has branch-points, on the ξ-axis, an analytical formula for $h(\zeta)$ may be difficult to construct, as $g_x(\xi)$ and $g_y(\xi)$ are discontinuous, $h(\zeta)$ still exists, however, although the evaluation of the integrals which arise, by means of residues, is then not possible. It is proposed to examine (2.532) in more detail, elsewhere, as an illustration of this.

When $h(\zeta)$ can be formulated, its expansions near its poles will be useful, or better those of $h(\zeta)/(\zeta^2+1)^2$, the poles are $\zeta = \iota$, and those of $\overline{g(\bar{\zeta})}$, in the upper half of the ζ-plane. Near $\zeta = \iota$, the expansion will be found to be (*cf.* (2.431)),

$$h(\zeta)/(\zeta^2+1)^2 = H_{-2}/\delta^2 + H_{-1}/\delta + H_0 + H_1\delta + H_2\delta^2 + \ldots, \qquad (2.624)$$

the coefficients being given by

$$H_{-2} = -\tfrac{1}{4}h(\iota), \qquad H_{-1} = -\tfrac{1}{4}\{\iota h(\iota) + h'(\iota)\},$$
$$H_0 = \tfrac{1}{16}\{3h(\iota) - 4\iota h'(\iota) + 2h''(\iota)\},$$
$$H_1 = \tfrac{1}{48}\{6\iota h(\iota) + 9h'(\iota) - 6\iota h''(\iota) - 2h'''(\iota)\},$$

etc.

Near ζ_n, the expansion will be denoted by

$$S_{-1n}/\delta + S_{0n} + S_{1n}\delta + \dots, \qquad (2.625)$$

where the coefficients can be obtained from the expansion (TAYLOR's series) for $g(\zeta)$, and (2.619)—but both (2.625) and (2.624) are usually more conveniently found from the explicit expression for $h(\zeta)/(\zeta^2+1)^2$.

(2.7) *Geometrical Properties of the Curve.*—The whole geometry of the curve is implied in the transformation formula. The tangent at the point ξ makes an angle ϕ with the x-axis given by

$$\tan\phi = f_y(\xi)/f_x(\xi), \qquad (2.71)$$

and the element of arc is

$$d\sigma = \sqrt{f(\xi)\,\overline{f(\xi)}}/(\xi^2+1)^2 \,.\, d\xi. \qquad (2.72)$$

From these, the radius and centre of curvature are easily obtained.

The area enclosed by the curve is

$$A = \tfrac{1}{2}\oint (x\,dy - y\,dx). \qquad (2.731)$$

Now

$$\begin{aligned}\bar{z}\,dz &= (x-\iota y)(dx+\iota\,dy)\\ &= (x\,dx+y\,dy)+\iota(x\,dy-y\,dx),\end{aligned}$$

so that

$$A = -\tfrac{1}{2}\iota\oint \bar{z}\,dz, \qquad (2.732)$$

since

$$\oint (x\,dx+y\,dy) = [\tfrac{1}{2}(x^2+y^2)] = 0.$$

On the boundary,

$$\bar{z}\,dz = \overline{g(\xi)}\,f(\xi)\,.\,d\xi/(\xi^2+1)^3,$$

so that, with

$$k(\xi) = \overline{g(\xi)}\,.\,f(\xi), \qquad (2.733)$$

we have

$$\begin{aligned}A &= -\tfrac{1}{2}\iota\int_{\infty}^{-\infty} k(\xi)/(\xi^2+1)^3\,.\,d\xi\\ &= \tfrac{1}{2}\iota\int_{-\infty}^{\infty} k(\xi)/(\xi^2+1)^3\,.\,d\xi. \qquad (2.734)\end{aligned}$$

In general, $k(\xi)$ is $0(\xi^4)$ as $\xi\to\infty$, so that the integral (like most of those we shall have to consider) can be evaluated by residues, by considering a contour consisting of the ξ-axis and an infinite semicircle in the upper half of the ζ-plane, provided, of course, that $k(\zeta)$ has no branch-points.

For the centroid of the enclosed area, z_G, we have,

$$\begin{aligned}Az_G &= \iint (x+\iota y)\,dx\,dy = \oint (\tfrac{1}{2}x^2\,dy - \tfrac{1}{2}\iota y^2\,dx)\\ &= \oint \tfrac{1}{2}\{(x^2+y^2)\,dy - \iota(x^2+y^2)\,dx\}, \qquad (2.741)\end{aligned}$$

since

$$\oint y^2\, dy = 0 = \oint x^2\, dx,$$

so that

$$A z_G = -\tfrac{1}{2}\iota \oint (x^2 + y^2)(dx + \iota\, dy) = \tfrac{1}{2}\iota \int_{-\infty}^{\infty} \frac{h(\xi)}{(\xi^2+1)^2} \cdot \frac{f(\xi)}{(\xi^2+1)^2}\, d\xi. \quad (2.742)$$

The integrand vanishes at infinity to such an order that the integral along an infinite semicircle vanishes, and so we can evaluate (2.742) by residues. The poles of the integrand are ι and ζ_n. Near ι

$$h(\xi)/(\xi^2+1)^2 = H_{-2}/\delta^2 + H_{-1}/\delta + H_0 + H_1\delta + \dots,$$
$$f(\xi)/(\xi^2+1)^2 = F_{-2}/\delta^2 + F_0 + F_1\delta + \dots,$$

so the residue is

$$H_{-2}F_1 + H_{-1}F_0 + H_1F_{-2}.$$

Near ζ_n,

$$h(\xi)/(\xi^2+1)^2 = S_{-1n}/\delta + S_{0n} + \dots, \qquad f(\xi)/(\xi^2+1)^2 = z'_n + z''_n\delta + \dots,$$

so the residue is

$$S_{-1n}\, z'_n.$$

Combining these two results, we have, finally

$$A z_G = -\pi\, \{H_{-2}F_1 + H_{-1}F_0 + H_1F_{-2} + \Sigma\, S_{-1n} z'_n\}. \quad . \quad . \quad (2.743)$$

(3) *Fundamental Potential Distributions.*

(3.1) "*Thermometric Parameters.*"—From the hydrodynamical standpoint, the first potential distribution of importance is that for a fluid circulating (irrotationally) round a stationary cylindrical obstacle. If Γ is the "circulation," then the velocity potential-stream function must tend to infinity with z like $\iota\Gamma/2\pi \,.\, \log z$. The transformed function for the ζ-plane will then tend to infinity near $\zeta = \iota$ like $-\iota\Gamma/2\pi \,.\, \log(\zeta - \iota)$. This is represented by a line-vortex cutting the ζ-plane in $\zeta = \iota$, of strength $-\Gamma$. Since the ξ-axis is to be a stream-line, the method of images gives us the complete solution by introducing an equal and opposite vortex at $\zeta = -\iota$. Our potential function is, then,

$$w_\Gamma = \phi_\Gamma + \iota\psi_\Gamma = \frac{\iota\Gamma}{2\pi} \log \frac{\zeta + \iota}{\zeta - \iota}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.11)$$

where ϕ_Γ is the velocity potential, and ψ_Γ the stream function. This may also be written

$$\zeta = -\cot(\pi w_\Gamma/\Gamma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.12)$$

We have already seen analytical reasons for substituting $\zeta = \tan \tfrac{1}{2}t$, and doing so here we find

$$t = 2\pi w_\Gamma/\Gamma + \pi,$$

or

$$w_\Gamma = \Gamma t/2\pi - \tfrac{1}{2}\Gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.13)$$

The constant, $\frac{1}{2}\Gamma$, is without effect, so we may take

$$w_\Gamma = \Gamma t/2\pi, \qquad \phi_\Gamma = \Gamma r/2\pi, \qquad \psi_\Gamma = \Gamma s/2\pi, \quad . \; . \; . \; . \; . \; . \quad (3.14)$$

i.e., the curves $r =$ const., and $s =$ const. are the equipotentials and stream-lines in this case; this is the physical meaning of the t-substitution, and also of what we have already called the "fundamental net," in § (2.1).

Other interpretations of the solution are :—

In electrostatics, a charged cylinder in space, when $-\psi_\Gamma$ is the potential, and $\phi_\Gamma =$ const. the lines of force. The increase of ϕ_Γ in going once round the boundary is $-4\pi/\text{K}$ times the charge per unit length, K being the dielectric constant of the surrounding medium. If e denotes this charge, $e = \text{K}\Gamma/4\pi$, or $\Gamma = 4\pi e/\text{K}$.

In the two-dimensional flow of electricity, the boundary represents an electrode, $-\psi$ is the potential, and $\phi =$ const. the lines of flow; the total current flowing, i per unit length, is $-1/\tau$ times the increase of ϕ in going round the boundary, τ being the specific resistance of the surrounding medium, so that $i = \Gamma/\tau$, or $\Gamma = i\tau$. A similar interpretation is that for a plane current sheet.

In the two-dimensional flow of heat, ψ is temperature and $\phi =$ const. the lines of flow. The total quantity of heat flowing out per unit time per unit length is then given by $\text{Q} = k\Gamma$, or $\Gamma = \text{Q}/k$, k being the conductivity.

In Appendix I we give, in tabular form, the possible interpretations with the appropriate changes of meaning of the symbols, and in the body of the paper we shall note interpretations other than hydrodynamical in a few interesting cases only.

With the circulating liquid, the velocity along the boundary is given by $-\partial\phi/\partial l$, δl being an element of the boundary. Now $\delta l = -|f(\xi)|/(\xi^2+1)^2 \,.\, \delta\xi$; also on the boundary ψ vanishes. Thus

$$-\partial\phi/\partial l = (\xi^2+1)^2/|f(\xi)| \,.\, dw_\Gamma/d\xi \text{ and } dw_\Gamma/d\xi = \Gamma/\pi\,(\xi^2+1),$$

so

$$q = \frac{\Gamma}{\pi}\frac{(\xi^2+1)}{|f(\xi)|}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.15)$$

which becomes infinite when $f(\xi) = 0$, *i.e.*, at a salient angle.

Any two stream-lines may be considered as rigid boundaries, when we shall have liquid circulating in the space between two cylinders with parallel generators. In the electrostatic analogy to this, we have a condenser whose plates are parallel cylinders, and if these are the curves $s = s_1$ and $s = s_2$, the capacity per unit length is easily found to be $\text{K}/2\,(s_2 - s_1)$, by the use of the third of equations (3.14) (*cf.* D. M. WRINCH, *loc. cit.*).

(3.2) *Boundary as Obstacle in a Stream.*—To solve the problem of the disturbance to a uniform stream caused by the introduction of a cylindrical obstacle, we need a function w_U which tends to infinity with z like $\text{U}ze^{-i\gamma}$, U being the undisturbed velocity of the stream, coming from a direction inclined at γ to the x-axis, and of which the imaginary part is constant along the boundary. In the ζ-plane, by (2.427), w_U must tend to

$-\mathrm{UF}_{-2}e^{-\iota\gamma}/(\zeta - \iota)$ as $\zeta \to \iota$, and this represents a double source at $\zeta = \iota$. To make the ξ-axis a stream-line, we must introduce a conjugate double source at $\zeta = -\iota$, so that

$$w_{\mathrm{U}} = -\mathrm{U}\left\{\frac{\mathrm{F}_{-2}e^{-\iota\gamma}}{\zeta - \iota} + \frac{\bar{\mathrm{F}}_{-2}e^{\iota\gamma}}{\zeta + \iota}\right\}. \qquad (3.21)$$

Putting $-\mathrm{F}_{-2} = \nu e^{\iota\alpha}$ and $\zeta = \tan \frac{1}{2}t$, this gives

$$w_{\mathrm{U}} = \nu\mathrm{U}\left\{\sin(t - \alpha + \gamma) - \sin(\alpha - \gamma)\right\}, \qquad (3.22)$$

which agrees with the formula of LEATHEM.* The term $\sin(\alpha - \gamma)$ is without effect, so we may write

$$\phi_{\mathrm{U}} = \nu\mathrm{U}\cosh s \sin(r - \alpha + \gamma) \qquad (3.231)$$

$$\psi_{\mathrm{U}} = \nu\mathrm{U}\sinh s \cos(r - \alpha + \gamma), \qquad (3.232)$$

although it is convenient to use (3.21) still. Corresponding values of r and s are easily calculated from these, for a series of values of w_{U}, and will serve for *all* such distributions, so that a table or the equivalent graphs, such as those given in Appendix II, enables equipotentials and stream-lines to be speedily plotted on the "fundamental net"—preferably on tracing paper over a plot of the net. We also see that $\psi_{\mathrm{U}} = 0$ corresponds to the stream-line which divides and follows the boundary of the obstacle, its other portions, $r = \alpha - \gamma \pm \frac{1}{2}\pi$, being radial lines of the "fundamental net."

It is a simple step to the corresponding solution when the curve $s = s_0$ is taken as the boundary of an obstacle, for then

$$w_{\mathrm{U}} = \nu\mathrm{U}\left\{\sin(t - \alpha + \gamma - \iota s_0) - \sin(\alpha - \gamma + \iota s_0)\right\}, \qquad (3.24)$$

but the expression in terms of the *new* ζ and $f(\zeta)$ is still (3.21).

(3.3) *Vortex.*—A vortex of strength κ at the point z_1 necessitates an equal vortex at the corresponding point ζ_1. The method of images leads to an equal and opposite vortex at the conjugate point $\bar{\zeta}_1$. The motion due to these two, however, will give a "circulation" round the boundary, and to annul this we must introduce another vortex in the lower half of the ζ-plane, and also, of course, its image in the upper half. As there is to be only one vortex in the finite part of the z-plane, the last of these must be at $\zeta = \iota$; and the third, consequently, at $\zeta = -\iota$. The value of w_κ is thus

$$w_\kappa = \frac{\iota\kappa}{2\pi}\log\frac{(\zeta - \zeta_1)(\zeta + \iota)}{(\zeta - \bar{\zeta}_1)(\zeta - \iota)}. \qquad (3.31)$$

(3.4) *Source.*—The change from a vortex to a source cannot be made by a simple interchange of ϕ and ψ, since the boundary conditions are not then satisfied. With a source of strength m at z_1, we must have an equal source at the corresponding point ζ_1,

* 'Phil. Mag.,' vol. 35, pp. 119–130 (1918).

and, since the flow is to infinity in the z-plane, an equal sink at $\zeta = \iota$. To make the ξ-axis a stream-line, we must have an equal sink at the image of the last point, $\zeta = -\iota$. This gives

$$w_m = -\frac{m}{2\pi}\log\frac{(\zeta-\zeta_1)(\zeta-\bar{\zeta}_1)}{(\zeta-\iota)(\zeta+\iota)}. \quad \ldots \ldots \quad (3.41)$$

(3.5) *Doublet.*—The field due to a doublet is obtained by putting an equal doublet at the corresponding point, and a conjugate doublet at its image. If μ is the moment and α the inclination of its axis, in the ζ-plane, we have

$$w_\mu = \mu\left\{\frac{e^{\iota\alpha}}{\zeta-\zeta_1}+\frac{e^{-\iota\alpha}}{\zeta-\bar{\zeta}_1}\right\}. \quad \ldots \ldots \quad (3.51)$$

In applications other than hydrodynamical, we may require a solution whose real part is zero (or constant) on the boundary. Then

$$w'_\mu = \mu\left\{\frac{e^{\iota\alpha}}{\zeta-\zeta_1}-\frac{e^{-\iota\alpha}}{\zeta-\bar{\zeta}_1}\right\}. \quad \ldots \ldots \quad (3.52)$$

In these cases, since there is no total outflow, there is no singularity at $\zeta = \iota$.

(3.6) *General Potential Distribution.*—The most general potential distribution we shall consider in detail is that including circulation (Γ), a stream (U, γ), vortices (strengths $\kappa_1, \kappa_2, \ldots$, at points $\zeta = \alpha_1, \alpha_2, \ldots$), and sources (strengths $m_1, m_2, \ldots$, at points $\beta_1, \beta_2, \ldots$). Combining the solutions already found we have

$$\begin{aligned} w = {} & \frac{\iota\Gamma}{2\pi}\log\frac{\zeta+\iota}{\zeta-\iota} - U\left\{\frac{F_{-2}e^{-\iota\gamma}}{\zeta-\iota}+\frac{\bar{F}_{-2}e^{\iota\gamma}}{\zeta+\iota}\right\} \\ & + \Sigma\frac{\iota\kappa}{2\pi}\log\frac{(\zeta-\alpha)(\zeta+\iota)}{(\zeta-\bar{\alpha})(\zeta-\iota)} \\ & - \Sigma\frac{m}{2\pi}\log\frac{(\zeta-\beta)(\zeta-\bar{\beta})}{(\zeta+\iota)(\zeta-\iota)}. \quad \ldots \ldots \quad (3.61) \end{aligned}$$

If v represents the (vector) fluid velocity,

$$\bar{v} = -\frac{dw}{dz} = -\frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz},$$

or

$$dw/d\zeta = -\bar{v}z'. \quad \ldots \ldots \quad (3.62)$$

Now

$$\begin{aligned} \frac{dw}{d\zeta} = {} & \frac{\iota\Gamma}{2\pi}\left\{\frac{1}{\zeta+\iota}-\frac{1}{\zeta-\iota}\right\} + U\left\{\frac{F_{-2}e^{-\iota\gamma}}{(\zeta-\iota)^2}+\frac{\bar{F}_{-2}e^{\iota\gamma}}{(\zeta+\iota)^2}\right\} \\ & + \Sigma\frac{\iota\kappa}{2\pi}\left\{\frac{1}{\zeta-\alpha}+\frac{1}{\zeta+\iota}-\frac{1}{\zeta-\bar{\alpha}}-\frac{1}{\zeta-\iota}\right\} \\ & - \Sigma\frac{m}{2\pi}\left\{\frac{1}{\zeta-\beta}+\frac{1}{\zeta-\bar{\beta}}-\frac{1}{\zeta+\iota}-\frac{1}{\zeta-\iota}\right\}. \quad \ldots \quad (3.63) \end{aligned}$$

For future use we shall need the expansions of $dw/d\zeta$ valid near the poles ι, α_n, and β_m.

(1) Near ι, with $\zeta = \iota + \delta$, we find

$$\frac{dw}{d\zeta} = \mathrm{U}\,\frac{\mathrm{F}_{-2}e^{-\iota\gamma}}{\delta^2} - \frac{\iota\,(\Gamma + \Sigma\kappa + \iota\Sigma m)}{2\pi\delta}$$
$$+ \left\{-\mathrm{U}\,\frac{\overline{\mathrm{F}}_{-2}e^{\iota\gamma}}{4} + \frac{\Gamma + \Sigma\kappa - \iota\Sigma m}{4\pi} + \Sigma\,\frac{\iota\kappa}{2\pi}\left(\frac{1}{\iota - \alpha} - \frac{1}{\iota - \overline{\alpha}}\right) - \Sigma\,\frac{m}{2\pi}\left(\frac{1}{\iota - \beta} + \frac{1}{\iota - \overline{\beta}}\right)\right\}$$
$$+ \delta\left\{-\mathrm{U}\,\frac{\overline{\mathrm{F}}_{-2}e^{\iota\gamma}}{4} + \frac{\iota\,(\Gamma + \Sigma\kappa - \iota\Sigma m)}{8\pi} - \Sigma\,\frac{\iota\kappa}{2\pi}\left(\frac{1}{(\iota - \alpha)^2} - \frac{1}{(\iota - \overline{\alpha})^2}\right) + \Sigma\,\frac{m}{2\pi}\left(\frac{1}{(\iota - \beta)^2} + \frac{1}{(\iota - \overline{\beta})^2}\right)\right\}. \quad (3.641)$$

(2) Near α_n. This expansion is expressible in terms of the velocity of the vortex, since " vortex lines move with the fluid." If v_n is this velocity,

$$\overline{v}_n = -\underset{z \to z_n}{\mathrm{Lt}}\,\frac{d}{dz}\left\{w - \frac{\iota\kappa_n}{2\pi}\log\,(z - z_n)\right\}$$
$$= -\frac{1}{z'}\,\mathrm{Lt}\left\{\frac{dw}{d\zeta} - \frac{\iota\kappa_n}{2\pi}\,\frac{d}{d\delta}\log\,(z'\delta + \tfrac{1}{2}z''\delta^2 + \ldots)\right\}$$
$$= -\frac{1}{z'}\,\mathrm{Lt}\left\{\frac{dw}{d\zeta} - \frac{\iota\kappa_n}{2\pi}\left(\frac{1}{\delta} + \frac{1}{2}\,\frac{z''_n}{z'_n} + \ldots\right)\right\},$$

so that, near α_n,

$$\frac{dw}{d\zeta} = \frac{\iota\kappa}{2\pi\delta} - z'_n\overline{v}_n + \frac{\iota\kappa_n}{4\pi}\,\frac{z''_n}{z'_n} + \ldots . \quad (3.642)$$

(3) Similarly, near β_m, if v_m is the velocity at z_m due to $\{w + m/2\pi\,.\log\,(z - z_m)\}$, which is finite,

$$\frac{dw}{d\zeta} = -\frac{m}{2\pi\delta} - z'_m\overline{v}_m - \frac{m}{2\pi}\,\frac{z''_m}{z'_m} + \ldots . \quad (3.643)$$

Similar results will be required for $\partial\phi/\partial t$, or rather for the function of ζ which reduces to $\partial\phi/\partial t$ on the boundary; and since $\partial\psi/\partial t = 0$ on the boundary,* this function is no other than $\partial w/\partial t$. We shall restrict ourselves here to the motion of the vortices (the acceleration of a *moving* boundary will be considered later) as the only case of hydrodynamical importance, although an examination of the effects of a varying source might prove interesting; this latter may form the subject of a future investigation. We also neglect variations of vortex strength, which are in practice small compared with the motions of the vortices.

* Since ψ is identically zero there.

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On the boundary, then, by (3.61)

$$\frac{\partial \phi}{\partial t} = -\Sigma \frac{\iota\kappa}{2\pi}\left(\frac{\dot{\alpha}}{\xi - \alpha} - \frac{\dot{\bar{\alpha}}}{\xi - \bar{\alpha}}\right), \quad \ldots \ldots \quad (3.651)$$

where $\dot{\alpha}_n$ is the velocity of the n-th vortex in the ζ-plane. We must have, also,

$$\dot{\alpha}_n = \frac{d\alpha_n}{dz}\frac{dz_n}{dt} = \frac{v_n}{z'_n}. \quad \ldots \ldots \quad (3.652)$$

Replacing ξ by ζ in (3.651), and denoting the resulting function by $(\partial\phi/\partial t)$, we see that $(\partial\phi/\partial t)$ has poles at $\alpha_1, \alpha_2, \ldots$. Near α_n the expansion is, with $\zeta = \alpha_n + \delta$,

$$\left(\frac{\partial \phi}{\partial t}\right) = -\frac{\iota\kappa v_n}{2\pi z'_n \delta} + \ldots, \quad \ldots \ldots \quad (3.653)$$

in which the result (3.652) has been incorporated.

(4) *Fluid Pressure on Boundary.*

If p is the pressure at any point on the boundary the resultant force (per unit length of cylinder) on an element of the boundary of length δl is $p \,.\, \delta l$, along the inward-drawn normal; this may be written $\iota p\,\delta z$, δz being the (complex) element of the boundary. Consequently, if X and Y are the components of the resultant force due to the fluid pressure, we have

$$\mathrm{X} + \iota\mathrm{Y} = \iota \oint p\,dz = \iota \int_{\infty}^{-\infty} p\frac{dz}{d\xi}d\xi$$

$$= -\iota \int_{-\infty}^{\infty} p\frac{dz}{d\xi}d\xi, \quad \ldots \ldots \quad (4.11)$$

where the path of integration is the real axis, indented above any branch-points or poles on this axis, if necessary.

Again,

$$p = \rho\,(\partial\phi/\partial t - \tfrac{1}{2}q^2), \quad \ldots \ldots \quad (4.21)$$

q being the resultant velocity at any point, so that it will be convenient to divide the integral into two, one depending on q (which will have interpretations other than hydrodynamical), and the other on $\partial\phi/\partial t$ (which is peculiar to the hydrodynamical interpretations). Using an obvious notation,

$$\mathrm{X}_q + \iota\mathrm{Y}_q = \tfrac{1}{2}\iota\rho \int_{-\infty}^{\infty} q^2\frac{dz}{d\xi}d\xi. \quad \ldots \ldots \quad (4.22)$$

$$\mathrm{X}_\phi + \iota\mathrm{Y}_\phi = -\iota\rho \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial t}\cdot\frac{dz}{d\xi}d\xi. \quad \ldots \ldots \quad (4.23)$$

Taking first the q integral, we have, generally,

$$q = \left|\frac{dw}{dz}\right| = \frac{dw}{d\xi}\cdot\left|\frac{d\xi}{dz}\right|, \quad \ldots \quad (4.31)$$

since on the boundary, $dw/d\xi$ is purely real (ψ being constant). Thus

$$X_q + \iota Y_q = \tfrac{1}{2}\iota\rho\int_{-\infty}^{\infty}\left(\frac{dw}{d\xi}\right)^2 \frac{dz}{d\xi}\Big/\left|\frac{dz}{d\xi}\right|^2 . d\xi$$

$$= \tfrac{1}{2}\iota\rho\int_{-\infty}^{\infty}\left(\frac{dw}{d\xi}\right)^2 \Big/\frac{d\bar{z}}{d\xi} . d\xi,$$

remembering that, since ξ is real, $\left|\frac{dz}{d\xi}\right|^2 = \frac{dz}{d\xi}\cdot\frac{d\bar{z}}{d\xi}$. Taking the conjugate quantities (again remembering that $dw/d\xi$ is real) we have

$$X_q - \iota Y_q = -\tfrac{1}{2}\iota\rho\int_{-\infty}^{\infty}\left(\frac{dw}{d\xi}\right)^2 \Big/\frac{dz}{d\xi} . d\xi. \quad \ldots \quad (4.321)$$

This may also be written

$$X_q - \iota Y_q = \tfrac{1}{2}\iota\rho\oint\left(\frac{dw}{dz}\right)^2 dz, \quad \ldots \quad (4.322)$$

a result known as BLASIUS' Theorem. If w has no singularities, in the relevant portion of the z-plane, the contour may be deformed into the circle at infinity, and in this form the result is used in aerodynamical calculations.*

The integral (4.321) may be evaluated by employing a contour consisting of the real axis and an infinite semicircle in the upper half of the ζ-plane. Provided the integral round the semicircle vanishes, the result will be $2\pi\iota$ times the sum of the residues of the integrand in the upper half of the ζ-plane, for the integrand is a uniform function of ζ in this region. Taking the general value of w given by (3.61) and the corresponding value of $dw/d\zeta$ given by (3.63), we see that $dw/d\zeta$ is of order $1/\zeta^2$ at infinity, so that, unless $\zeta = \infty$ corresponds to a corner of the boundary, the integral over the infinite semicircle will vanish.

The poles of the integrand are ι, α_n, and β_m. The expansions of $dw/d\zeta$ and $dz/d\zeta$ near these have been given, and using them we have :—

Near ι,

$$\left(\frac{dw}{d\zeta}\right)^2 = \frac{(UF_{-2}e^{-\iota\gamma})^2}{\delta^4} - \frac{\iota UF_{-2}e^{-\iota\gamma}\{\Gamma + \Sigma\kappa + \iota\Sigma m\}}{\pi\delta^3} + \ldots$$

by (3.641), and

$$\frac{1}{dz/d\zeta} = \frac{\delta^2}{F_{-2}} + O(\delta^4)$$

* *Cf.* GLAUERT, 'Elements of Aerofoil and Airscrew Theory,' Camb. Univ. Press (1927), p. 81.

by (2.436), so that the residue is

$$-\iota U e^{-\iota\gamma}\{\Gamma + \Sigma\kappa + \iota\Sigma m\}/\pi. \qquad (4.323)$$

Near α_n,

$$\left(\frac{dw}{d\zeta}\right)^2 = -\frac{\kappa_n^2}{4\pi^2\delta^2} + \frac{\iota\kappa_n}{\pi\delta}\left\{\frac{\iota\kappa_n}{4\pi}\frac{z''_n}{z'_n} - \bar{v}_n z'_n\right\}$$

by (3.642), and

$$\frac{1}{dz/d\zeta} = \frac{1}{z'_n + z''_n\delta + \ldots} = \frac{1}{z'_n} - \frac{z''_n}{z'^2_n}\delta + \ldots,$$

so that the residue is

$$\frac{\kappa_n^2}{4\pi^2}\frac{z''}{z'^2} + \frac{\iota\kappa_n}{\pi z'_n}\left\{\frac{\iota\kappa_n}{4\pi}\frac{z''_n}{z'_n} - \bar{v}_n z'_n\right\} = -\frac{\iota\kappa_n}{\pi}\bar{v}_n. \qquad (4.324)$$

Near β_m,

$$\left(\frac{dw}{d\zeta}\right)^2 = \frac{m^2}{4\pi^2\delta^2} + \frac{m}{\pi\delta}\left(\frac{m}{4\pi}\frac{z''_m}{z'_m} + \bar{v}_m z'_m\right)$$

by (3.643), and

$$\frac{1}{dz/d\zeta} = \frac{1}{z'_m} - \frac{z''_m}{z'^2_m}\delta,$$

so that the residue is

$$\frac{m}{\pi}\bar{v}_m. \qquad (4.325)$$

Collecting residues, we have

$$X_q - \iota Y_q = -\tfrac{1}{2}\iota\rho \times 2\pi\iota\,[-\iota U e^{-\iota\gamma}\{\Gamma + \Sigma\kappa + \iota\Sigma m\} - \Sigma\iota\kappa\bar{v} + \Sigma m\bar{v}]/\pi \;. \qquad (4.326)$$

or, taking conjugate quantities, and reducing,

$$X_q + \iota Y_q = \rho\,[\iota U e^{\iota\gamma}\{\Gamma + \Sigma\kappa - \iota\Sigma m\} + \Sigma\iota\kappa v + \Sigma m v]. \qquad (4.327)$$

In the absence of vortices or sources, this reduces to $\iota\rho\Gamma U e^{\iota\gamma}$, which is a force perpendicular to the direction of the stream; this is the LANCHESTER-PRANDTL formula for aerofoil lift, and for the MAGNUS effect. If we consider the electrostatic analogy we have a charged cylinder in a uniform field, and the force then reduces to Fe/K parallel to the field; this is an almost obvious result, and it makes the aerodynamic formula equally so.

The only case in which the integral round the infinite semicircle does not vanish is that when $\zeta = \infty$ corresponds to a cusp $(\alpha_\infty = \pi)$* of the z-boundary. For, if $\zeta = \infty$ corresponds to a corner, $dz/d\zeta \to K\zeta^{-(2+\alpha_\infty/\pi)}$, by (2.429), while $dw/d\zeta \to A/\zeta^2$. Consequently, with $\zeta = Re^{\iota\theta}$,

$$\left(\frac{dw}{d\zeta}\right)^2 \Big/ \frac{dz}{d\zeta} \,.\, d\zeta \to \iota A^2 (Re^{\iota\theta})^{\alpha_\infty/\pi - 1}/K \,.\, d\theta,$$

* α_∞ having the same meaning as in § 2.4.

so that, unless $\alpha_\infty = \pi$ (and α_∞ cannot be $> \pi$), the integral over the semicircle vanishes. If $\alpha_\infty = \pi$, its value is

$$\pi\iota A^2/K. \quad \ldots \quad (4.328)$$

This means, of course, that there is a finite force due to the streaming round the cusp, and the existence of such a force was pointed out by the author in 1918,* and its value calculated for the boundary and field there discussed. The corresponding forces for the cusps not corresponding to $\xi = \infty$ are, so to speak, taken by the integral in its stride, in virtue of the indentations above the zeroes of $f(\xi)$.

We also see that the resultant force depends upon the *motions* of the vortices, and as their motion will alter the value of w, we *cannot* neglect the $\partial\phi/\partial t$ term in the pressure equation. We proceed to calculate its effect.

From (4.23)

$$X_\phi + \iota Y_\phi = -\iota\rho\int_{-\infty}^{\infty} \frac{\partial\phi}{\partial t}\frac{dz}{d\xi}d\xi.$$

We can easily show that the integral round the infinite semicircle vanishes (unless $\alpha_\infty = -\pi$, which is impossible) and so the value of the integral is $2\pi\iota \times$ the sum of the residues in the upper half of the ζ-plane. The poles are α_n and ι.

Near α_n,

$$\left(\frac{\partial\phi}{\partial t}\right) = -\frac{\iota\kappa v_n}{2\pi z'_n\delta} + \ldots,$$

and so the residue is

$$-\iota\kappa_n v_n/2\pi. \quad \ldots \quad (4.411)$$

Near ι

$$\frac{dz}{d\xi} = \frac{F_{-2}}{\delta^2} + F_0 \ \ldots,$$

$$\left(\frac{\partial\phi}{\partial t}\right) = -\Sigma\frac{\iota\kappa}{2\pi}\left(\frac{v}{z'(\iota-\alpha)} - \frac{\bar{v}}{\bar{z}'(\iota-\bar{\alpha})}\right) + \Sigma\frac{\iota\kappa}{2\pi}\left(\frac{v}{z'(\iota-\alpha)^2} - \frac{\bar{v}}{\bar{z}'(\iota-\bar{\alpha})^2}\right)\delta + \ldots,$$

so the residue is

$$\frac{\iota F_{-2}}{2\pi}\Sigma\kappa\left(\frac{v}{z'(\iota-\alpha)^2} - \frac{\bar{v}}{\bar{z}'(\iota-\bar{\alpha})^2}\right). \quad \ldots \quad (4.412)$$

Combining these, we have

$$X_\phi + \iota Y_\phi = -\Sigma\iota\kappa\rho v + \iota\rho F_{-2}\Sigma\kappa\left(\frac{v}{z'(\iota-\alpha)^2} - \frac{\bar{v}}{\bar{z}'(\iota-\bar{\alpha})^2}\right). \quad (4.42)$$

Finally, adding the results of (4.33) and (4.42),

$$X + \iota Y = \rho\left[\iota U e^{\iota\gamma}(\Gamma + \Sigma\kappa - \iota\Sigma m) + \Sigma m v + F_{-2}\Sigma\kappa\left(\frac{v}{z'(\iota-\alpha)^2} - \frac{\bar{v}}{\bar{z}'(\iota-\bar{\alpha})^2}\right)\right], \quad (4.51)$$

which is the final formula for the force on the obstacle, per unit length, due to the fluid motion considered, and which was mentioned in a recent paper by the author.†

We next calculate the moment of the forces due to the fluid pressure on the boundary,

* 'Phil. Mag.,' 6, vol. 35, pp. 396–404 (1918).

† 'Roy. Soc. Proc.,' A, vol. 119, p. 146 (1928).

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about the origin, which will complete our knowledge of these forces. The moment of the force on an element δz of the boundary is $p\,(x\,\delta x + y\,\delta y)$ which is seen to be the real part of $p\,(x + \iota y)\,(\delta x - \iota\,\delta y)$, so

$$\mathrm{M} = \mathfrak{R} \oint pz\, d\bar{z}.^* \qquad (4.611)$$

Dividing M into two parts, due to the two terms in the pressure equation, we have first

$$\mathrm{M}_q = -\tfrac{1}{2}\rho\mathfrak{R} \oint q^2 z\, d\bar{z} = \tfrac{1}{2}\rho\mathfrak{R} \int_{-\infty}^{\infty} \left(\frac{dw}{d\xi}\right)^2 \left|\frac{d\xi}{dz}\right|^2 z \frac{d\bar{z}}{d\xi} d\xi$$

$$= \tfrac{1}{2}\rho\mathfrak{R} \int_{-\infty}^{\infty} \left(\frac{dw}{d\xi}\right)^2 \frac{z}{dz/d\xi} d\xi. \qquad (4.621)$$

This can also be written

$$-\tfrac{1}{2}\rho\mathfrak{R} \oint \left(\frac{dw}{dz}\right)^2 z\, dz, \qquad (4.622)$$

the form used in aerodynamics†; in the absence of singularities, this can be taken round an infinite circle.

To evaluate (4.621) by residues, we first notice that the integrand is of order $\zeta^{-(2-\alpha_\infty/\pi}$ at infinity, so that, unless $\alpha_\infty = \pi$, the integral round the infinite semicircle vanishes. In the exceptional case, we have already shown the existence of a finite force at the cusp, and this can be allowed for by (4.328). The poles of the integrand are ι, α_n, and β_m.

Near ι, by (3.631)

$$dw/d\zeta = \mathrm{P}/\delta^2 + \mathrm{Q}/\delta + \mathrm{R} + \dots,$$

where

$$\mathrm{P} = \mathrm{UF}_{-2}e^{-\iota\gamma}, \qquad \mathrm{Q} = -\iota\,(\Gamma + \Sigma\kappa + \iota\Sigma m)/2\pi$$

$$\mathrm{R} = -\tfrac{1}{4}\mathrm{U}\overline{\mathrm{F}}_{-2}e^{\iota\gamma} + \frac{\Gamma + \Sigma\kappa - \iota\Sigma m}{4\pi} + \Sigma\frac{\iota\kappa}{2\pi}\left(\frac{1}{\iota - \alpha} - \frac{1}{\iota - \bar{\alpha}}\right) - \Sigma\frac{m}{2\pi}\left(\frac{1}{\iota - \beta} + \frac{1}{\iota - \bar{\beta}}\right).$$

Also, by (2.428)

$$\frac{z}{dz/d\zeta} = -\delta\left\{1 + \frac{\mathrm{G}_0}{\mathrm{F}_{-2}}\delta - \frac{2\mathrm{F}_0}{\mathrm{F}_{-2}}\delta^2 + \dots\right\},$$

so that the residue is

$$\frac{2\mathrm{F}_0}{\mathrm{F}_{-2}}\mathrm{P}^2 - \frac{2\mathrm{G}_0\mathrm{PQ}}{\mathrm{F}_{-2}} - (2\mathrm{RP} + \mathrm{Q}^2)$$

$$= 2\mathrm{U}^2 e^{-2\iota\gamma}\mathrm{F}_{-2}\mathrm{F}_0 + \iota\mathrm{G}_0\mathrm{U}e^{-\iota\gamma}(\Gamma + \Sigma\kappa + \iota\Sigma m)/\pi$$

$$+ \tfrac{1}{2}\mathrm{U}^2\mathrm{F}_{-2}\overline{\mathrm{F}}_{-2} - 2\mathrm{UF}_{-2}e^{-\iota\gamma}\left\{\frac{\Gamma + \Sigma\kappa - \iota\Sigma m}{4\pi} + \Sigma\frac{\iota\kappa}{2\pi}\left(\frac{1}{\iota - \alpha} - \frac{1}{\iota - \bar{\alpha}}\right)\right.$$

$$\left. - \Sigma\frac{m}{4\pi}\left(\frac{1}{\iota - \beta} + \frac{1}{\iota - \bar{\beta}}\right)\right\} + \frac{(\Gamma + \Sigma\kappa + \iota\Sigma m)^2}{4\pi^2}. \qquad (4.631)$$

* $\mathfrak{R}$ denoting "the real part of."

† GLAUERT, *loc. cit.*

Near α_n, by (3.632)

$$\left(\frac{dw}{d\zeta}\right)^2 = -\frac{\kappa^2{}_n}{4\pi^2\,\delta^2} + \frac{\iota\kappa_n}{\pi\delta}\left(\frac{\iota\kappa_n}{4\pi}\frac{z''{}_n}{z'{}_n} - \overline{v}_n z'{}_n\right)$$

and

$$\frac{z}{dz/d\zeta} = \frac{z_n}{z'{}_n} + \delta\left(\frac{z'^2{}_n - z_n z''{}_n}{z'^2{}_n}\right) + \dots,$$

so the residue is

$$-\frac{\kappa^2{}_n}{4\pi^2} - \frac{\iota\kappa_n}{\pi} z_n\overline{v}_n. \qquad (4.632)$$

Similarly, near β_m, the residue is

$$\frac{m^2}{4\pi^2} + \frac{m}{\pi} z_m\overline{v}_m. \qquad (4.633)$$

Collecting these results, we have

$$\begin{aligned} M_q = \rho\mathfrak{R}\Big[& 2\iota\pi U^2 F_{-2}\,(F_0 e^{-2\iota\gamma} + \tfrac{1}{4}\overline{F}_{-2}) - G_0 U e^{-\iota\gamma}(\Gamma + \Sigma\kappa + \iota\Sigma m) \\ & -\iota U F_{-2} e^{-\iota\gamma}\left\{\tfrac{1}{2}(\Gamma + \Sigma\kappa - \iota\Sigma m) + \Sigma\iota\kappa\left(\frac{1}{\iota - \alpha} - \frac{1}{\iota - \overline{\alpha}}\right) - \Sigma m\left(\frac{1}{\iota - \beta} + \frac{1}{\iota - \overline{\beta}}\right)\right\} \\ & + \frac{\iota}{4\pi}\{(\Gamma + \Sigma\kappa + \iota\Sigma m)^2 - \Sigma\kappa^2 + \Sigma m^2\} + \Sigma\kappa z'\overline{v} + \iota\Sigma m z'\overline{v}\Big]. \end{aligned} \qquad (4.641)$$

The term in $F_{-2}\overline{F}_{-2}$ will contribute nothing, and can be omitted.

The G_0 term can be removed by taking moments about $z = G_0$, for if M'_q is the moment about this point,

$$M'_q = M_q - \mathfrak{R}\,\iota\,(X_q - \iota Y_q)\,G_0, \qquad (4.642)$$

and, by (4.326),

$$-\iota G_0\,(X_q - \iota Y_q) = -\rho G_0\,\{U e^{-\iota\gamma}(\Gamma + \Sigma\kappa + \iota\Sigma m) + \Sigma\kappa\overline{v} + \iota\Sigma m\overline{v}\}. \quad (4.643)$$

Using this, we remove the G_0 term in (4.641), the only other alteration being that we replace z by $(z - G_0)$ in the terms $\Sigma\kappa\overline{v}z$ and $\Sigma m\overline{v}z$, which is not a complication.

For translation only, we have

$$M_q = \rho\mathfrak{R}\,(2\pi\iota F_{-2} F_0 e^{-2\iota\gamma}). \qquad (4.644)$$

For the $\partial\phi/\partial t$ term we have

$$M_\phi = \rho\mathfrak{R}\oint\frac{\partial\phi}{\partial t} z\,d\overline{z} = -\rho\mathfrak{R}\oint_{-\infty}^{\infty}\frac{\partial\phi}{\partial t}\frac{g(\xi)\,\overline{f(\xi)}}{(1+\xi)^3}\,d\xi. \qquad (4.65)$$

The integration of this by residues involves a knowledge of the poles of $\overline{f(\xi)}$, and so cannot be simply expressed; it can, however, easily be effected in special cases.

It is also possible to obtain the moment by means of another formula, by using

$$x\,dx + y\,dy = \tfrac{1}{2}d\,(r^2) = \tfrac{1}{2}\frac{d}{d\xi}\left(\frac{h(\xi)}{(1+\xi^2)^2}\right)d\xi.$$

Then,

$$M = -\int_{-\infty}^{\infty} \tfrac{1}{2}p \frac{d}{d\xi}\left\{\frac{h(\xi)}{(1+\xi^2)^2}\right\} d\xi$$

$$= -\left[\tfrac{1}{2}p \frac{h(\xi)}{(1+\xi^2)^2}\right]_{-\infty}^{\infty} + \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{dp}{d\xi}\frac{h(\xi)}{(1+\xi^2)^2} d\xi. \quad \ldots \quad (4.661)$$

The integrated term vanishes, since it has the same value at $+\infty$ as at $-\infty$, the boundary being closed, and p being single-valued. Thus

$$M = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{dp}{d\xi}\frac{h(\xi)}{(1+\xi^2)^2} d\xi, \quad \ldots \quad (4.662)$$

which can be evaluated by residues when $h(\xi)$ is known; again there are poles of the integrand where $\overline{g(\bar{\zeta})}$—and consequently $\overline{f(\bar{\zeta})}$—has poles.

(5) *Boundary in Motion.*

(5.1) *Uniform Translation.*—The solution for the streaming past an obstacle at rest can be converted into that for the motion of the obstacle in a fluid at rest at infinity by the subtraction of the term $Uze^{-\iota\gamma}$, U being the velocity and γ the direction of motion.

(5.2) "*Impulse.*"—The "impulse" in such a case is, when Γ, κ, and m all vanish,

$$I = -\rho\oint \phi\iota dz = \iota\rho\int_{-\infty}^{\infty} \phi\frac{dz}{d\xi} d\xi, \quad \ldots \quad (5.21)$$

being the resultant of an impulsive pressure $\rho\phi$ applied to the fluid at the boundary. For the case in question, the velocity potential-stream function is

$$w = -U\left\{\frac{F_{-2}e^{-\iota\gamma}}{\zeta-\iota} + \frac{\bar{F}_{-2}e^{\iota\gamma}}{\zeta+\iota}\right\} - Uze^{-\iota\gamma}, \quad \ldots \quad (5.22)$$

and ϕ is the real part of this. The bracketed portion is purely real on the boundary, so it is a part of ϕ. The real part of the other term is $-U(x\cos\gamma + y\sin\gamma)$, so we may write the impulse as

$$I = -\iota\rho\int_{-\infty}^{\infty} U\left(\frac{F_{-2}e^{-\iota\gamma}}{\xi-\iota} + \frac{\bar{F}_{-2}e^{\iota\gamma}}{\xi+\iota}\right)\frac{f(\xi)}{(1+\xi^2)^2} d\xi$$

$$+ \iota\rho\oint U(x\cos\gamma + y\sin\gamma)(dx + \iota dy). \quad \ldots \quad (5.231)$$

The first integrand gives zero when taken round the infinite semicircle, and its only pole is $\zeta = \iota$. The expansion near $\zeta = \iota$ is

$$U\left\{\frac{F_{-2}e^{-\iota\gamma}}{\delta} + \frac{\bar{F}_{-2}e^{\iota\gamma}}{2\iota}\left(1 - \frac{\delta}{2\iota} + \ldots\right)\right\}\left\{\frac{F_{-2}}{\delta^2} + F_0 + \ldots\right\}$$

and so the residue is

$$UF_{-2}\left(\tfrac{1}{4}\bar{F}_{-2}e^{\iota\gamma} + F_0 e^{-\iota\gamma}\right).$$

The contribution to the "impulse" is thus

$$2\pi\rho UF_{-2}\left(\tfrac{1}{4}\bar{F}_{-2}e^{\iota\gamma} + F_0 e^{-\iota\gamma}\right). \quad \ldots \quad (5.232)$$

For the other integral we have $\oint x\,dx = 0 = \oint y\,dy$, and $\oint x\,dy = A = -\oint y\,dx$, A being the area enclosed by the boundary. The value is thus,

$$\iota\rho U(\iota A\cos\gamma - A\sin\gamma) = -\rho AUe^{\iota\gamma}. \quad \ldots \quad (5.233)$$

This is seen to be — the momentum (per unit length) of the liquid the boundary would enclose, moving as a solid with velocity U in the direction γ. Thus (5.232) may be taken as giving the impulse, if it be understood to include also the momentum of the fluid the boundary would contain. A similar interpretation has been given by LEATHEM to his simple formula for the impulse,* and our result is derivable from his.

Writing the impulse in the form

$$2\pi\rho U\left(\tfrac{1}{4}F_{-2}\bar{F}_{-2}e^{\iota\gamma} + F_{-2}F_0 e^{-\iota\gamma}\right), \quad \ldots \quad (5.234)$$

we see that, unless the second term vanishes, the impulse vector is not generally parallel to the velocity vector. For directions of the velocity given by

$$F_{-2}F_0 e^{-\iota\gamma_1} = De^{\iota\gamma_1},$$

where D is real, the two vectors will have the same direction, *i.e.*, if,

$$F_{-2}F_0 = De^{2\iota\gamma_1}. \quad \ldots \quad (5.241)$$

There are thus two principal directions of motion, at right angles, when the impulse and velocity vectors are parallel, and if, in any motion, I_A and I_B are the components of the impulse in these directions,

$$I_A = 2\pi\rho U\left(\tfrac{1}{4}F_{-2}\bar{F}_{-2} + D\right)\cos(\gamma - \gamma_1), \quad \ldots \quad (5.242)$$

$$I_B = 2\pi\rho U\left(\tfrac{1}{4}F_{-2}\bar{F}_{-2} - D\right)\sin(\gamma - \gamma_1). \quad \ldots \quad (5.243)$$

For the kinetic energy of the motion (including that of the enclosed liquid, of course), we have

$$T = \tfrac{1}{2}\{I_A U\cos(\gamma - \gamma_1) + I_B U\sin(\gamma - \gamma_1)\}$$

$$= \pi\rho U^2\{\tfrac{1}{4}F_{-2}\bar{F}_{-2} + D\cos 2(\gamma - \gamma_1)\}, \quad \ldots \quad (5.251)$$

and the principal inertia coefficients are

$$2\pi\rho\{\tfrac{1}{4}F_{-2}\bar{F}_{-2} \pm |F_{-2}F_0|\}. \quad \ldots \quad (5.252)$$

* 'Phil. Trans.,' A, vol. 215, p. 439 (1915).

The moment, L, of the impulse is given by

$$\mathrm{L} = -\oint \rho\phi\,(x\,dx + y\,dy) = -\oint \rho\phi\, d\,(\tfrac{1}{2}r^2)$$

$$= -\,[\tfrac{1}{2}\rho\phi r^2] + \tfrac{1}{2}\rho\oint r^2\,d\phi = -\,\tfrac{1}{2}\rho\int_{-\infty}^{\infty}\frac{h(\xi)}{(1+\xi^2)^2}\frac{d\phi}{d\xi}\,d\xi, \quad . \quad . \quad (5.26)$$

the integrated part vanishing. In any actual case this integral can be evaluated when $h(\xi)$ is known.

The actual impulse is then given in magnitude and direction by I, but its line of action is at a distance L/I from the origin. This is relative to the body, but the line moves with the body in space. The rate of change of the moment of the impulse about a *fixed* point is then

$$\mathrm{U}\,(\mathrm{I_X}\sin\gamma - \mathrm{I_Y}\cos\gamma) = -\,\mathfrak{I}\,(\mathrm{IU}e^{-\iota\gamma}).* \quad . \quad . \quad . \quad . \quad . \quad (5.271)$$

To maintain the motion, a couple of magnitude equal to this must be applied, from which we see that the moment of the fluid pressures is

$$\mathrm{M}_q = \mathfrak{I}\,(\mathrm{IU}e^{-\iota\gamma}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.272)$$

being equal and opposite to this. Comparing this with (4.644) and using (5.232) we see that the results agree, and we can write

$$\mathrm{M}_q = 2\pi\rho\mathrm{U}^2\mathrm{D}\sin 2\,(\gamma - \gamma_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.273)$$

(5.3) *Force and Moment due to Fluid Pressures.*—It is easy to show from the fundamental equations that the pressure difference between two points is the same when the boundary is moving in fluid at rest at infinity as it is when the fluid streams past the boundary, the relative motion being the same. Consequently (4.51) and (4.641) with (4.65), still give the resultant force and moment due to fluid pressures, provided we now take v_n and v_m to be relative to the boundary.

If the obstacle is accelerated, however, we have an extra contribution to $\partial\phi/\partial t$, and therefore to the force and moment. The value of this addition to $\partial\phi/\partial t$, on the boundary is

$$-\,\dot{\mathrm{U}}\left(\frac{\mathrm{F}_{-2}e^{-\iota\gamma}}{\xi-\iota} + \frac{\overline{\mathrm{F}}_{-2}e^{\iota\gamma}}{\xi+\iota}\right) + \iota\mathrm{U}\dot{\gamma}\left(\frac{\mathrm{F}_{-2}e^{-\iota\gamma}}{\xi-\iota} - \frac{\overline{\mathrm{F}}_{-2}e^{\iota\gamma}}{\xi+\iota}\right)$$

$$-\,\dot{\mathrm{U}}\,(x\cos\gamma + y\sin\gamma) - \mathrm{U}\dot{\gamma}\,(y\cos\gamma - x\sin\gamma). \quad . \quad . \quad (5.31)$$

The resultant force due to this is

$$\iota\rho\int_{-\infty}^{\infty}\left[\dot{\mathrm{U}}\left\{\frac{\mathrm{F}_{-2}e^{-\iota\gamma}}{\xi-\iota} + \frac{\overline{\mathrm{F}}_{-2}e^{\iota\gamma}}{\xi+\iota}\right\} - \iota\mathrm{U}\dot{\gamma}\left\{\frac{\mathrm{F}_{-2}e^{-\iota\gamma}}{\xi-\iota} - \frac{\overline{\mathrm{F}}_{-2}e^{\iota\gamma}}{\xi+\iota}\right\}\right]\frac{dz}{d\xi}\,d\xi$$

$$-\,\iota\rho\oint\{\dot{\mathrm{U}}\,(x\cos\gamma + y\sin\gamma) + \mathrm{U}\dot{\gamma}\,(y\cos\gamma - x\sin\gamma)\}\,(dx - \iota\,dy). \quad . \quad (5.32)$$

* $\mathfrak{I}$ denoting "the imaginary part of."

To evaluate the first integral, by residues, as has been done in many previous cases, is easy, there being one pole only, $\xi = \iota$, and the residue there is

$$- \{\tfrac{1}{4}F_{-2}\bar{F}_{-2}(\dot{U} + \iota U\dot{\gamma})\, e^{\iota\gamma} + F_{-2}F_0(\dot{U} - \iota U\dot{\gamma})\, e^{-\iota\gamma}\},$$

so that the contribution to the resultant force is

$$- 2\pi\rho\, \{\tfrac{1}{4}F_{-2}\bar{F}_{-2}(\dot{U} + \iota U\dot{\gamma})\, e^{\iota\gamma} + F_{-2}F_0(\dot{U} - \iota U\dot{\gamma})\, e^{-\iota\gamma}\}. \quad . \quad . \quad (5.321)$$

The other integral gives

$$- \iota\rho\dot{U}A(\iota\cos\gamma - \sin\gamma) - \iota\rho UA\dot{\gamma}(-\cos\gamma - \iota\sin\gamma)$$
$$= \rho A(\dot{U} + \iota U\dot{\gamma})\, e^{\iota\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.322)$$

which is seen to be the force necessary to produce the acceleration in the enclosed fluid, $(\dot{U} + \iota U\dot{\gamma})\, e^{\iota\gamma}$ being the total acceleration of the boundary.

The applied force which will produce the acceleration is equal and opposite to the sum of (5.321) and (5.322). If, however, as before, we consider the boundary to be full of liquid, the force is

$$2\pi\rho\, \{\tfrac{1}{4}F_{-2}\bar{F}_{-2}(\dot{U} + \iota U\dot{\gamma})\, e^{\iota\gamma} + F_{-2}F_0(\dot{U} - \iota U\dot{\gamma})\, e^{-\iota\gamma}\}. \quad . \quad . \quad . \quad . \quad (5.34)$$

and this is, as it should be, the time rate of change of the corresponding "impulse."

We may insert here another treatment of the force on the boundary, and incidentally obtain a result for future use.

Using the suffix 1 to denote the boundary, and 2 to denote any closed curve outside,

$$X_p + \iota Y_p = \iota \oint_1 p \,.\, dz = - \tfrac{1}{2}\rho \oint_1 \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}(\iota dx - dy).$$

Now

$$\oint_1 \tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}(\iota dx - dy) - \oint_2 \tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}(\iota dx - dy)$$
$$= \iint \tfrac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} dx\, dy$$
$$= \iint \left\{\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x} + \iota\frac{\partial\phi}{\partial y}\right) + \frac{\partial\phi}{\partial y}\cdot\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x} + \iota\frac{\partial\phi}{\partial y}\right)\right\} dx\, dy$$
$$= - \oint_1 \frac{\partial\phi}{\partial n}\left(\frac{\partial\phi}{\partial x} + \iota\frac{\partial\phi}{\partial y}\right) ds + \oint_2 \frac{\partial\phi}{\partial n}\left(\frac{\partial\phi}{\partial x} + \iota\frac{\partial\phi}{\partial y}\right) ds - \iint \left(\frac{\partial\phi}{\partial x} + \iota\frac{\partial\phi}{\partial y}\right)\nabla^2\phi \,.\, dx\, dy,$$

where n is the outward drawn normal in each case. The double integral vanishes if ϕ has no singularities between 1 and 2. Also, in any case to be considered, ϕ will be of order at most $1/r$ at infinity, so that if we take 2 to be the circle at infinity, the line integral round it will vanish. Thus

$$\oint_1 \tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}\iota\,dz=-\oint_1\frac{\partial\phi}{\partial n}\left(\frac{\partial\phi}{\partial x}+\iota\frac{\partial\phi}{\partial y}\right)ds. \quad . \quad . \quad . \quad . \quad (5.35)$$

For a stationary boundary, $\partial\phi/\partial n$ vanishes identically so that $X_p+\iota Y_p=0$. For uniform translation, $\phi=-Ux+\phi_1$. ϕ_1, which is the same function as for the stationary boundary, is, however, referred to axes moving with the boundary, so that the $\partial\phi/\partial t$ term must be inserted. Then

$$\frac{\partial\phi}{\partial t}=\frac{\partial\phi_1}{\partial t}=-U\frac{\partial\phi_1}{\partial x},$$

and

$$p=\rho\left[\frac{\partial\phi}{\partial t}-\tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}\right]=\rho\left[-U\frac{\partial\phi_1}{\partial x}-\tfrac{1}{2}\left(-U+\frac{\partial\phi_1}{\partial x}\right)^2-\tfrac{1}{2}\left(\frac{\partial\phi_1}{\partial y}\right)^2\right]$$

$$=-\tfrac{1}{2}\rho\left\{U^2+\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial y}\right)^2\right\},$$

so

$$X+\iota Y=\iota\oint p\,.\,dz=-\tfrac{1}{2}\rho\oint\left\{U^2+\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial y}\right)^2\right\}\iota\,.\,dz$$

$$=-\tfrac{1}{2}\rho\iota U^2[z]+\rho\oint\frac{\partial\phi_1}{\partial n}\left(\frac{\partial\phi_1}{\partial x}+\iota\frac{\partial\phi_1}{\partial y}\right)ds$$

$$=0.$$

(5.4) *Rotating Boundary.*—To solve the hydrodynamical problem of the flow due to a rotating boundary, we have to determine a potential function $w_\omega=\phi_\omega+\iota\psi_\omega$, such that ψ_ω reduces to $\frac{1}{2}\omega r^2$ on the boundary, ω being the angular velocity, and r the distance of the point from the axis of rotation, with the additional conditions that w_ω is finite and single-valued in the relevant region, including infinity. We need therefore a function of ζ with its imaginary part having the prescribed value on the ξ-axis, and finite and single-values in the upper half of the ζ-plane. Now it is known that if $\psi(\xi)$ satisfies certain conditions as to integrability, the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\psi(\xi)}{\xi-\zeta}\,d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.41)$$

has its imaginary part tending to $\psi(\xi)$ as ζ approaches the ξ-axis from above, and has no singularity in the upper half of the ζ-plane. In our case, $\psi(\xi)=\frac{1}{2}\omega r^2$, which is evidently both finite and continuous in the boundary, so that the conditions for integrability are fulfilled. We may, moreover, without loss of generality, take the centre of rotation to be the origin in the z-plane, so that $r^2=|z|^2$. Our formula is then

$$w_\omega=\frac{\omega}{2\pi}\int_{-\infty}^{\infty}\frac{|z|^2}{\xi-\zeta}\,d\xi$$

$$=\frac{\omega}{2\pi}\int_{-\infty}^{\infty}\frac{h(\xi)}{(1+\xi^2)^2}\,\frac{d\xi}{\xi-\zeta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.42)$$

Taking a contour consisting of the real axis and an infinite semicircle c in the upper half of the plane (where ξ is now considered to be a complex variable), and restricting ourselves to the case where $h(\zeta)$ has no branch-points, we have

$$w_\omega + \frac{\omega}{2\pi}\int_c \frac{|z_\infty|^2\, d\xi}{\xi - \zeta} = \iota\,\omega \times \text{(sum of residues in upper half-plane).}$$

Now z_∞ is finite, so the integral on the left is $\frac{1}{2}\omega\,|z_\infty|^2$ which is constant, and so has no effect upon the flow. We may therefore take the quantity on the right as w_ω.

The poles of the integrand are ι, ζ, and those of $h(\xi)$, denoted in ζ_n in § (2.6). Near ι,

$$h(\xi)/(1+\xi^2)^2 = \mathrm{H}_{-2}/\delta^2 + \mathrm{H}_{-1}/\delta + \mathrm{H}_0 + \ldots$$

$$1/(\xi - \zeta) = -1/(\zeta - \iota) - \delta/(\zeta - \iota)^2 - \ldots,$$

so the residue is

$$-\frac{\mathrm{H}_{-2}}{(\zeta - \iota)^2} - \frac{\mathrm{H}_{-1}}{(\zeta - \iota)}. \qquad (5.431)$$

Near ζ, the residue is

$$\frac{h(\zeta)}{(1+\zeta^2)^2}. \qquad (5.432)$$

Near ζ_n,

$$h(\xi)/(1+\xi^2)^2 = \mathrm{S}_{-1n}/\delta + \ldots,$$

$$1/(\xi - \zeta) = -1/(\zeta - \zeta_n) - \ldots,$$

so the residue is

$$-\frac{\mathrm{S}_{-1n}}{\zeta - \zeta_n}. \qquad (5.433)$$

Consequently,

$$w_\omega = \iota\,\omega\left\{\frac{h(\zeta)}{(1+\zeta^2)^2} - \frac{\mathrm{H}_{-2}}{(\zeta-\iota)^2} - \frac{\mathrm{H}_{-1}}{\zeta - \iota} - \Sigma\,\frac{\mathrm{S}_{-1n}}{\zeta - \zeta_n}\right\}. \qquad (5.434)$$

When $h(\zeta)$ has branch-points—and it will have them at points on the ξ-axis corresponding to corners of the boundary, and may have them elsewhere—(5.42) still holds and can be evaluated, but (5.434) is no longer true.

On the boundary we have

$$\psi_\omega = \tfrac{1}{2}\omega h(\xi)/(1+\xi^2)^2, \qquad (5.435)$$

so that, again on the boundary,

$$\phi_\omega = \frac{\iota\,\omega}{2}\left\{\frac{h(\xi)}{(1+\xi^2)^2} - \frac{2\mathrm{H}_{-2}}{(\xi-\iota)^2} - \frac{2\mathrm{H}_{-1}}{\xi - \iota} - \Sigma\,\frac{2\mathrm{S}_{-1n}}{\xi - \zeta_n}\right\}. \qquad (5.436)$$

Remembering that $h(\xi)$ is real for real values of ξ, it is easy to see that this expression is real when $h(\xi)$ is rational.

(5.5) *Impulse in Rotational Motion.*—Although the motion of the boundary is a rotation, it does not follow that the impulse reduces to a couple; we calculate therefore the impulse and its moment about the origin. As in (5.21)

$$\mathrm{I}_\omega = -\rho\oint\phi\,.\,\iota\,.\,dz = \iota\rho\int_{-\infty}^{\infty}(\phi)\frac{dz}{d\xi}\,d\xi. \quad \ldots\ldots \quad (5.511)$$

The integral round the infinite semicircle vanishes, and the poles of the integrand are ι and ζ_n. Near ι,

$$(\phi) = \tfrac{1}{2}\iota\,\omega\left\{-\frac{\mathrm{H}_{-2}}{\delta^2}-\frac{\mathrm{H}_{-1}}{\delta}+\left(2\Sigma\frac{\mathrm{S}_{-1n}}{\zeta_n-\iota}+\mathrm{H}_0\right)+\left(2\Sigma\frac{\mathrm{S}_{-1n}}{(\zeta_n-\iota)^2}+\mathrm{H}_1\right)\delta+\ldots\right\},$$

$$\frac{dz}{d\xi} = \frac{\mathrm{F}_{-2}}{\delta^2}+\mathrm{F}_0+\mathrm{F}_1\delta+\ldots,$$

so the residue is

$$\tfrac{1}{2}\iota\,\omega\left\{-\mathrm{H}_{-2}\mathrm{F}_1-\mathrm{H}_{-1}\mathrm{F}_0+\mathrm{F}_{-2}\left(2\Sigma\frac{\mathrm{S}_{-1n}}{(\zeta_n-\iota)^2}+\mathrm{H}_1\right)\right\}. \quad \ldots \quad (5.512)$$

Near ζ_n,

$$\phi = -\tfrac{1}{2}\iota\,\omega\mathrm{S}_{-1n}/\delta+\ldots,\quad dz/d\xi = z'_n+\ldots,$$

so the residue is

$$-\tfrac{1}{2}\iota\,\omega\mathrm{S}_{-1n}z'_n. \quad \ldots\ldots\ldots \quad (5.513)$$

Thus the formula for I is

$$\mathrm{I}_\omega = \iota\pi\omega\rho\,\{\mathrm{H}_{-2}\mathrm{F}_1+\mathrm{H}_{-1}\mathrm{F}_0-\mathrm{H}_1\mathrm{F}_{-2}-2\mathrm{F}_{-2}\Sigma\mathrm{S}_{-1n}/(\zeta_n-\iota)^2+\Sigma\mathrm{S}_{-1n}z'_n\}\ \ldots \quad (5.514)$$

It can be easily verified that a change of centre of rotation, to (x_1, y_1) say—equivalent to a change of origin—alters H_{-2}, H_1, and S_{-1n}, linearly in x_1, y_1, and leaves H_{-1} unchanged. It is thus possible to find one, and only one, centre of rotation such that I vanishes.

The formula for I can also be obtained as follows:

$$\mathrm{I} = -\oint\rho\phi\iota\,.\,dz = -\iota\rho\oint(w_\omega-\tfrac{1}{2}\omega r^2)\,dz$$

$$= -\iota\rho\oint w_\omega\,dz-\iota\,\omega\rho\mathrm{A}z_{\mathrm{G}}$$

or

$$\mathrm{I}+\iota\,\omega\rho\mathrm{A}z_{\mathrm{G}} = -\iota\rho\oint w_\omega\,dz. \quad \ldots\ldots \quad (5.521)$$

But w_ω has no singularities in the z-plane, so the integral is $2\pi\iota\mathrm{W}_{-1}$, where W_{-1} is the coefficient of $1/z$ in the expansion of w_ω valid near infinity, so that

$$\mathrm{I}+\iota\,\omega\rho\mathrm{A}z_{\mathrm{G}} = 2\pi\rho\mathrm{W}_{-1}. \quad \ldots\ldots \quad (5.522)$$

Also $\iota\omega\rho A z_G$ is the momentum of the liquid that the boundary would enclose, moving as a rigid body, so that this again agrees with LEATHEM's result.* Moreover, near $z = \infty$, $\zeta = \iota + \delta$, and $z \to -F_{-2}/\delta$, so that $W_{-1} = -F_{-2} \times$ coeff. of δ in w_w

$$= -\iota\omega F_{-2}\{H_1 + \Sigma S_{-1n}/(\zeta_n - \iota)^2\}. \quad . \quad . \quad . \quad . \quad (5.523)$$

Using the value of Az_G from (2.743), we reproduce (5.514).

For the moment of the impulse we have,

$$L_\omega = -\rho \oint \phi d\left(\tfrac{1}{2}r^2\right) = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} \phi \frac{d}{d\xi}\left(\frac{h(\xi)}{(1+\xi^2)^2}\right) d\xi$$

$$= \tfrac{1}{4}\iota\omega\rho \int_{-\infty}^{\infty} \left\{\frac{h(\xi)}{(1+\xi^2)^2} - \frac{2H_{-2}}{(\xi-\iota)^2} - \frac{2H_{-1}}{(\xi-\iota)} - 2\Sigma\frac{S_{-1n}}{\xi-\zeta_n}\right\} \frac{d}{d\xi}\left(\frac{h(\xi)}{(1+\xi^2)^2}\right) d\xi$$

$$= \tfrac{1}{4}\iota\omega\rho \left\{\left[\frac{1}{2}\frac{h(\xi)}{(1+\xi^2)^2}\right]_{-\infty}^{\infty} - 2\int\left(\frac{H_{-2}}{(\xi-\iota)^2} + \frac{H_{-1}}{(\xi-\iota)} + \Sigma\frac{S_{-1n}}{\xi-\zeta_n}\right)\frac{d}{d\xi}\left(\frac{h(\xi)}{(1+\xi^2)^2}\right)\right\} d\xi. \quad (5.53)$$

The integrated portion vanishes as the boundary is closed. The integrand of the remaining integral is of order not greater than $1/\zeta^2$ at infinity, so that the integral is $2\pi\iota \times$ (sum of residues). The poles are ι and ζ_n. At ι, the residue is

$$2H_{-2}H_2 + H_{-1}H_1 + H_{-1}\Sigma S_{-1n}/(\zeta - \iota)^2 + 2H_{-2}\Sigma S_{-1n}/(\zeta_n - \iota)^3,$$

and at ζ_n,

$$S_{-1n}S_{1n} + S_{-1n}\{2H_{-2}/(\zeta_n - \iota)^3 + H_{-1}/(\zeta_n - \iota)^2 + \Sigma'_m S_{-1m}/(\zeta_m - \zeta_n)^2\}.$$

So

$$L = \pi\omega\rho\{2H_{-2}H_2 + H_{-1}H_1 + \Sigma S_{-1n}S_{1n} + 4H_{-2}\Sigma S_{-1n}/(\zeta_n - \iota)^3$$
$$+ 2H_{-1}\Sigma S_{-1n}/(\zeta_n - \iota)^2 + 2\Sigma\Sigma' S_{-1n}S_{-1m}/(\zeta_m - \zeta_n)^2\} \quad . \quad . \quad (5.54)$$

where $\Sigma\Sigma'$ is to include, once only, all different pairs of suffixes m, n. The equivalent moment of inertia is then L/ω.

(5.6) *Resultant Force.*—If the centre of rotation is such that the impulse does not reduce to a couple, then, since this is constant relative to the body, *i.e.*, *turns with it*, a force $\iota\omega I$ must be applied to the body to maintain the motion; the resultant of the fluid pressures is then equal and opposite to this, and so is $-\iota\omega I$.

This can be shown quite generally as follows. Since w_ω as obtained refers to axes moving with the body, the $\partial\phi/\partial t$ term will be present in the pressure equation. In fact,

$$\frac{\partial\phi}{\partial t} = -\omega\frac{\partial\phi}{\partial\theta} = \omega\left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right), \quad . \quad . \quad . \quad . \quad (5.61)$$

and so

$$p = \rho\left\{\omega\left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right) - \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \tfrac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2\right\}. \quad . \quad . \quad . \quad (5.62)$$

* 'Phil. Trans.,' A, vol. 215, pp. 439–487 (1915).

Therefore

$$X + \iota Y = \oint p\iota \, . \, dz = \rho \oint \left[\omega \left(y \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial y} \right) - \tfrac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 - \tfrac{1}{2} \left(\frac{\partial \phi}{\partial y} \right)^2 \right] (\iota \, dx - dy)$$

$$= \rho \oint \Big[\omega \Big(\iota y \frac{\partial \phi}{\partial x} dx + \iota y \frac{\partial \phi}{\partial y} dy - \iota x \frac{\partial \phi}{\partial y} dx - \iota y \frac{\partial \phi}{\partial y} dy + x \frac{\partial \phi}{\partial y} dy$$

$$+ x \frac{\partial \phi}{\partial x} dx - y \frac{\partial \phi}{\partial x} dy - x \frac{\partial \phi}{\partial x} dx \Big) + \frac{\partial \phi}{\partial n} \left(\frac{\partial \phi}{\partial x} + \iota \frac{\partial \phi}{\partial y} \right) ds \Big]$$

where we have used (5.35)

$$= \rho \oint \Big[\omega \Big\{ (x + \iota y) \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right) - \left(\frac{\partial \phi}{\partial x} + \iota \frac{\partial \phi}{\partial y} \right) (x \, dx + y \, dy) \Big\}$$

$$+ \frac{\partial \phi}{\partial n} \left(\frac{\partial \phi}{\partial x} + \iota \frac{\partial \phi}{\partial y} \right) ds \Big] .$$

Now

$$\omega \, (x \, dx + y \, dy) = \frac{\partial \psi}{\partial s} ds = \frac{\partial \phi}{\partial n} ds ,$$

therefore

$$X + \iota Y = \rho \omega \oint (x + \iota y) \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right)$$

$$= \rho \omega \oint z \, d\phi = \rho \omega \left\{ [z\phi] - \oint \phi \, dz \right\} .$$

As ϕ is single-valued, the integrated part vanishes, so

$$X + \iota Y = - \rho \omega \oint \phi \, dz = - \iota \omega \left\{ - \rho \oint \phi \, . \, \iota dz \right\}$$

$$= - \iota \omega I. \qquad \qquad (5.63)$$

In the case of unsteady motion, there must also be a force $\dot{\omega} \, (I_\omega / \omega)$, and a couple $\dot{\omega} \, (L_\omega / \omega)$ applied to produce the acceleration, so that the resultant of the fluid pressures is equal and opposite to these.

6. *An Illustrative Example.*

As the object of the foregoing has been to construct a " technique " for the solution of two-dimensional potential problems from a known transformation formula, it seems desirable to conclude with a pointed " Q.E.F." in the shape of an illustration of the application of the formulæ obtained to a particular case. That chosen is that of the circular arc, as being simple, but not too simple, and as one for which the author worked out results by more pedestrian methods some time since.

The $z - \zeta$ formula is given above (2.582), but this has the disadvantage—in dealing with rotation, that is—that the z-origin is the centre of the circle. Transferring the origin to the mid-point of the arc, we have (see fig. 6)

$$z = -r\frac{2(a-1)\zeta}{(\zeta-\iota)(\zeta+a\iota)} = -2r\frac{a-1}{a+1}\left\{\frac{1}{\zeta-\iota}+\frac{a}{\zeta+a\iota}\right\}, \quad . \quad . \quad . \quad (6.1)$$

where, if 2α is the angle of the arc, $a = (1+\sin\frac{1}{2}\alpha)(1-\sin\frac{1}{2}\alpha)$.

From this we obtain

$$\frac{dz}{d\zeta} = 2r\frac{a-1}{a+1}\left\{\frac{1}{(\zeta-\iota)^2}+\frac{a}{(\zeta+a\iota)^2}\right\}$$

$$= r\frac{2(a-1)(\zeta^2-a)}{(\zeta-\iota)^2(\zeta+a\iota)^2}, \quad . \quad . \quad (6.21)$$

whence

$$\mathrm{F}_{-2} = 2r\frac{a-1}{a+1} = \bar{\mathrm{F}}_{-2}, \quad \mathrm{F}_0 = -2r\frac{a-1}{(a+1)^3},$$

$$\mathrm{F}_1 = -4\iota r\frac{a(a-1)}{(a+1)^4}. \quad . \quad (6.22)$$

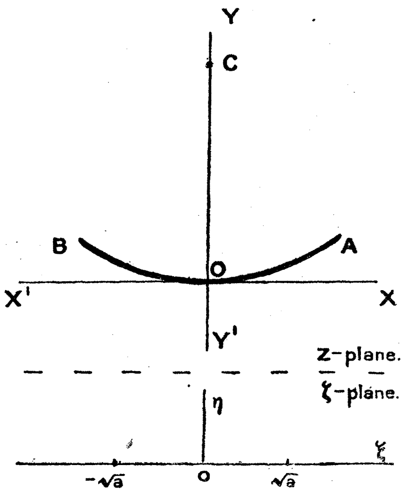

FIG. 6.

At the ends of the arc, $dz/d\zeta = 0$, so $\zeta = \pm\sqrt{a}$, leading to the above value of a in terms of α.

Then, by (2.617),

$$(\bar{z}) = -r\frac{2(a-1)\zeta}{(\zeta+\iota)(\zeta-a\iota)} = -2r\frac{a-1}{a+1}\left\{\frac{1}{\zeta+\iota}+\frac{a}{\zeta-a\iota}\right\}, \quad . \quad . \quad . \quad (6.3)$$

which has a pole at $\zeta = a\iota$ in the upper half of the ζ-plane.

Also, by (2.624)

$$(r^2) = r^2\frac{4(a-1)^2\zeta^2}{(\zeta^2+1)(\zeta^2+a^2)} = 2\iota r^2\frac{a-1}{a+1}\left\{\frac{1}{\zeta-\iota}-\frac{1}{\zeta+\iota}+\frac{a}{\zeta+a\iota}-\frac{a}{\zeta-a\iota}\right\}, \quad (6.41)$$

whence

$$\mathrm{H}_{-2} = 0, \quad \mathrm{H}_{-1} = 2\iota r^2\frac{a-1}{a+1}, \quad \mathrm{H}_0 = r^2\frac{3a^2+1}{(a+1)^2}, \quad \mathrm{H}_1 = -\iota r^2\frac{a^4+14a^2+1}{2(a-1)(a+1)}. \quad (6.42)$$

As might be concluded from (6.3), (r^2) has a pole at $\zeta = a\iota$, where

$$\mathrm{S}_{-1} = -2\iota r\frac{a(a-1)}{a+1}, \quad \mathrm{S}_0 = r^2\frac{a^2+3}{(a+1)^2}, \quad \mathrm{S}_1 = \iota r^2\frac{a^4+14a^2+1}{2a(a-1)(a+1)^3} \quad (6.43)$$

and

$$z' = -\frac{a+1}{2a(a-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.44)$$

From (2.733), using (6.1) and (6.3),

$$k(\zeta)/(\zeta^2+1)^3 = 4r^2 \frac{(a-1)^2 \zeta (\zeta^2 - a)}{(\zeta - \iota)(\zeta + a\iota)(\zeta + \iota)^2 (\zeta - a\iota)^2}, \quad . \quad . \quad . \quad (6.51)$$

the poles of which are ι and $a\iota$. It is easy to verify that the residues at these are $+1$ and -1 respectively, so

$$A = \tfrac{1}{2}\iota \int_{-\infty}^{\infty} \frac{k(\xi)\, d\xi}{(\xi^2+1)^3} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.52)$$

Formula (2.743) also gives zero—as it should when A is zero—and consequently does not allow us to find z_G.

The fundamental net, drawn in the previous paper,* and the solution for streaming past, require no further mention. The resultant force in a stream vanishes, but its moment does not. By (4.64)

$$\begin{aligned} M_q &= \rho \mathfrak{R} (2\pi\iota U^2 F_{-2} F_0 e^{-2\iota\gamma}) \\ &= \rho \mathfrak{R} \left(2\pi\iota U^2 \,.\, 2r \frac{a-1}{a+1} \,.\, 2r \frac{-a(a-1)}{(a+1)^3} e^{-2\iota\gamma}\right) \\ &= \frac{2\pi\rho U^2 r^2 a (a-1)^2}{(a+1)^4} \sin 2\gamma \\ &= \tfrac{1}{2}\pi\rho U^2 r^2 \sin^2 \tfrac{1}{2}\alpha \cos^2 \tfrac{1}{2}\alpha \sin 2\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.6) \end{aligned}$$

agreeing with the result in the previous paper.

The impulse in translational motion is, by (5.234),

$$\begin{aligned} I &= 2\pi\rho U (\tfrac{1}{4} F_{-2} \bar{F}_{-2} e^{\iota\gamma} + F_{-2} F_0 e^{-\iota\gamma}) \\ &= 2\pi\rho r^2 U \left\{ \left(\frac{a-1}{a+1}\right)^2 e^{\iota\gamma} - \frac{4a(a-1)^2}{(a+1)^4} e^{-\iota\gamma} \right\} \\ &= 2\pi\rho r^2 U \sin^2 \tfrac{1}{2}\alpha \{e^{\iota\gamma} - \cos^2 \tfrac{1}{2}\alpha \,.\, e^{-\iota\gamma}\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.71) \end{aligned}$$

This is in the direction of the velocity when $\gamma = 0$ or $= 90°$. The principal inertia coefficients are (5.242),

$$2\pi\rho r^2 \sin^2 \tfrac{1}{2}\alpha\, (1 \mp \cos^2 \tfrac{1}{2}\alpha),$$

i.e.,

$$I_x = 2\pi\rho r^2 U \sin^4 \tfrac{1}{2}\alpha \cos\gamma, \quad I_y = 2\pi\rho r^2 U \sin^2 \tfrac{1}{2}\alpha\, (1 + \cos^2 \tfrac{1}{2}\alpha) \sin\gamma. \quad . \quad . \quad (6.72)$$

The moment of the impulse is given by (5.26) as

$$L = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} (r^2) \frac{\partial\phi}{\partial\xi}\, d\xi$$

and

$$\phi = U \left(\frac{F_{-2}\, e^{-\iota\gamma}}{\zeta - \iota} + \frac{\bar{F}_{-2}\, e^{\iota\gamma}}{\zeta + \iota}\right) = -\,2rU \frac{a-1}{a+1} \,.\, \frac{2(\zeta\cos\gamma + \sin\gamma)}{\zeta^2+1},$$

* 'Phil. Mag.,' 6, vol. 35, p. 398 (1918).

so

$$L = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} 8r^3 U \frac{(a-1)^3}{a+1} \cdot \frac{\xi^2 \{(\xi^2-1)\cos\gamma + 2\xi \sin\gamma\}}{(\xi^2+1)^3 (\xi^2+a^2)} \, d\xi.$$

Evaluating this by residues (the poles are ι and $a\iota$), we find

$$L = -\pi\rho U r^3 \cdot \frac{2(a-1)^4}{(a+1)^4} \cos\gamma = -2\pi\rho U r^3 \sin^4 \frac{\alpha}{2} \cos\gamma. \quad \ldots \quad (6.73)$$

Now as all the impulsive pressures are normal, and therefore radial, the impulse must pass through the centre of the circle. The moment about the mid-point of the arc should therefore be $-I_x r$, and by (6.72) we see that this is so.

By (5.434) we obtain the solution for rotation,

$$w_\omega = \iota\omega \left\{ \frac{4r^2(a-1)^2\zeta^2}{(\zeta^2+1)(\zeta^2+a^2)} - \frac{2\iota r^2(a-1)}{(a+1)(\zeta-\iota)} + \frac{2\iota r^2(a-1)}{(a+1)(\zeta-a\iota)} \right\}$$

$$= 2\omega r \frac{a-1}{a+1} \left\{ \frac{1}{\zeta+\iota} - \frac{a}{\zeta+a\iota} \right\}. \quad \ldots \quad (6.81)$$

Formula (5.514) gives for the impulse

$$I_\omega = \iota\pi\omega\rho \left\{ 0 - 2\iota r^2 \frac{a-1}{a+1} \cdot 2r \frac{a(a-1)}{(a+1)^3} + \iota r^2 \frac{a^4+14a^2+1}{2(a-1)(a+1)^3} \cdot 2r \frac{a-1}{a+1} \right.$$
$$\left. - 4r \frac{a-1}{a+1} \cdot 2\iota r^2 \frac{a(a-1)}{(a+1)^3} + 2\iota r^2 \frac{a(a-1)}{a+1} \cdot r \frac{a+1}{2a(a-1)} \right\}$$
$$= -2\pi\omega\rho r^3 \left(\frac{a-1}{a+1}\right)^4 = -2\pi\omega\rho r^3 \sin^4 \frac{\alpha}{2}. \quad \ldots \quad (6.82)$$

If we realise that a translation parallel to the x axis with velocity $U = \omega r$ added to the rotation will give a revolution round the centre of the circle, which will not disturb the fluid, this result is checked by comparison with the first of (6.72).

For the moment of the impulse (and again the impulse should pass through the centre, so that we should have $L_\omega = -rI_\omega$) we have, by (5.54),

$$L_\omega = \pi\omega\rho \left\{ -2\iota r^2 \frac{a-1}{a+1} \cdot \iota r^2 \frac{a^4+14a^2+1}{2(a-1)(a+1)^3} - 2\iota r^2 \frac{a(a-1)}{a+1} \cdot \iota r^2 \frac{a^4+14a^2+1}{2a(a-1)(a+1)^3} \right.$$
$$\left. - 2 \cdot 2\iota r^2 \frac{a-1}{a+1} \cdot 2\iota r^2 \frac{a(a-1)}{(a+1)(a\iota-\iota)^2} \right\}$$
$$= 2\pi\rho\omega r^4 \left(\frac{a-1}{a+1}\right)^4 = 2\pi\omega\rho r^4 \sin^4 \tfrac{1}{2}\alpha, \quad \ldots \quad (6.83)$$

which is equal to $-rI_\omega$ as expected. The equivalent moment of inertia of the fluid is

$$2\pi\rho r^4 \sin^4 \tfrac{1}{2}\alpha. \quad \ldots \quad (6.84)$$

In conclusion, the thanks of the author are due to Mr. J. P. Clatworthy, B.Sc., who kindly read through a draft of this paper, and to whose valuable suggestions the treatment at the ends of §§ 5.3 and 5.6, not involving the transformation formula, is due.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

APPENDIX I.—Symbols, etc., and their Interpretations.

	Hydrodynamics.	Electrostatics.	Electric current (*a*).*	Electric current (*b*).*	Steady heat flow.
Boundary	Rigid solid.	Conductor.	Electrode.	Hole.	Hot cylinder.
ϕ	Velocity potential.	Force function.	Current function.	Potential.	Flow function.
ψ	Stream function.	— Potential.	— Potential.	Current function.	Temperature.
Grad ϕ	— Fluid velocity.	—	—	E.M.F.	
Grad ψ	—	Electric force.	E.M.F.	—	Temperature gradient.
q	Resultant fluid velocity	Resultant electric force.	Resultant E.M.F.	Resultant E.M.F.	Resultant temperature gradient.
ρ	Fluid density.	Dielectric const. $K = 4\pi\rho$.	Specific resistance $\tau = 1/\rho$.		Conductivity, $k = \rho$.
ρq	Momentum/unit vol.	Polarisation; surface density.	Current density.		Heat flow/unit area.
$\frac{1}{2}\rho q^2$	K.E./unit vol.	Energy/unit vol. of dielectric.	$\frac{1}{2}$ × rate of heat generation/unit vol.		
$p_q = -\frac{1}{2}\rho q^2$	Pressure (velocity head)	Dielectric stress (on conductor).			
$p_\phi = \rho(\partial\phi/\partial t)$	Pressure (acceleration head)	—			
$X_q + \iota Y_q$	Mechanical force from p_q	Mechanical force.			
$X_\phi + \iota Y_\phi$	Mechanical force from p_ϕ	—			
M_q	Moment from p_q	Moment.			
M_ϕ	Moment from p_ϕ	—			
I	Impulse.				
L	Moment of impulse.				
U	Uniform stream; velocity.	Uniform field.	Uniform field.	Uniform field.	Uniform temperature gradient.
Γ	Circulation.	Charge, $e = K\Gamma/4\pi$.	Electrode current $i = \Gamma/\tau$	—	Total heat flow, $Q = k\Gamma$.
κ	Vortex; strength.	Line charge, $e = K\kappa/4\pi$.	Point (line) electrode, $i = \kappa/\tau$.	—	Line source; $Q = k\kappa$.
m	Source; strength.	—	—	Point (line) electrode; $i = m/\tau$.	—

* These apply to plane current sheets by substituting "unit area" for "unit volume," etc.

APPENDIX II.

Standard Graphs for Plotting Stream-Lines.—The fact that the solution of the problem of flow past an obstacle (equation (3.22)) is expressible in such general terms enables us to construct tables or graphs for the plotting of the equipotentials and stream-lines in all such cases, provided that the "fundamental net" has first been obtained. We give, in figs. 7 and 8, graphs of

$$\cosh s \cos r = \phi \qquad \text{(fig. 7)}$$

and

$$\sinh s \sin r = \psi \qquad \text{(fig. 8)}$$

which are essentially the conformally transformed equipotentials and stream-lines, the r-axis being the boundary. The unit in which r and s are expressed is $\pi/12$, which has been found convenient; one "quadrant" of r is all that is given, but it is clearly sufficient.

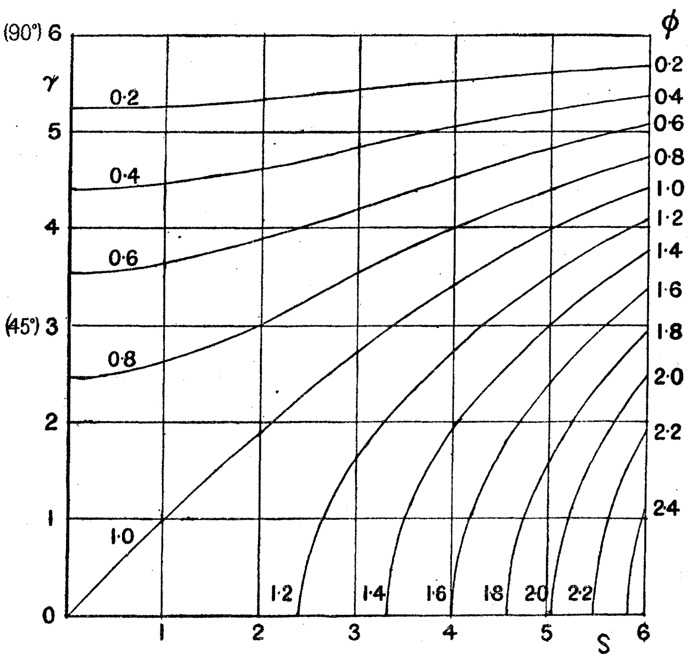

FIG. 7.

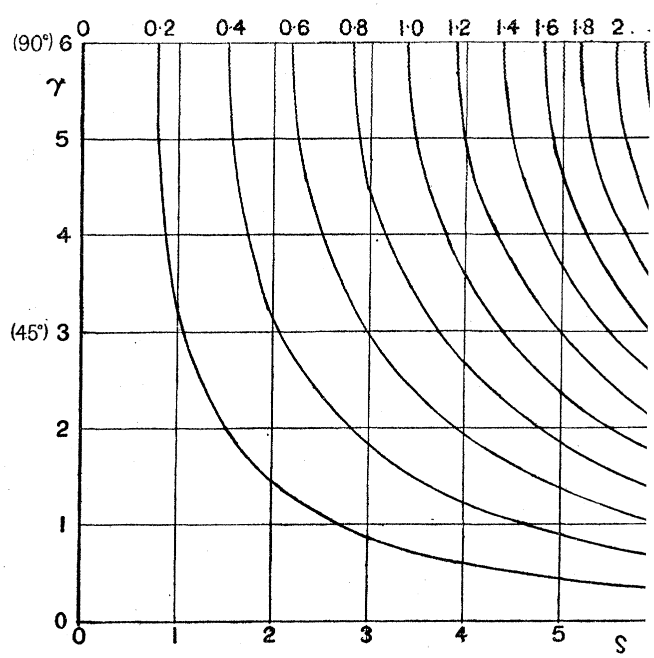

FIG. 8.